UNIVERSITÉ CHOUAIB DOUKKALI
FACULTÉ DES SCIENCES
DÉPARTEMENT DE MATHÉMATIQUES

Année Universitaire 2014-2015

SMP 3

Analyse 3 : Série 1

Exercice 1. Déterminer l'ensemble des points $z \in \mathbb{C}$, tels que :

1. $\frac{1}{2} < |z - i| < 1$.
2. $\frac{\pi}{4} < \mathrm{Arg} z < \frac{\pi}{3}$.
3. $(0 < \mathrm{Im} z < 1)$ ou $(-1 < \mathrm{Re} z < 1)$.

Exercice 2. Exprimer sous forme algébrique $z = x + iy$ les nombres complexes suivants :

a) $\mathrm{Log}(-e^2)$, b) $\mathrm{Log}(1 - i\sqrt{3})$,
c) $(-1)^i$, d) $i^{\mathrm{Log} i}$.

Exercice 3. Soient $z_1, z_2 \in \mathbb{C}^*$. Montrer que :

i) $\log(z_1.z_2) = \log z_1 + \log z_2$.
ii) $\mathrm{Log}(z_1.z_2) = \mathrm{Log} z_1 + \mathrm{Log} z_2$ ssi $-\pi < \mathrm{Arg} z_1 + \mathrm{Arg} z_2 \leq \pi$.
iii) Vérifier que si $|z| < 1$, alors $\mathrm{Log}(1 - z^2) = \mathrm{Log}(1 - z) + \mathrm{Log}(1 + z)$.

Exercice 4. Soit $z \in \mathbb{C}^*$, $z = x + iy$. Montrer que :

$$\mathrm{Arg} z = \begin{cases} \mathrm{Arctg}(\frac{y}{x}) & \text{si} \quad x > 0,\ y \in \mathbb{R}, \\ \pi + \mathrm{Arctg}(\frac{y}{x}) & \text{si} \quad x < 0,\ y \geq 0, \\ -\pi + \mathrm{Arctg}(\frac{y}{x}) & \text{si} \quad x < 0,\ y < 0, \\ \frac{\pi}{2} & \text{si} \quad x = 0,\ y > 0, \\ -\frac{\pi}{2} & \text{si} \quad x = 0,\ y < 0. \end{cases}$$

Exercice 5. Calculer les limites suivantes :

i) $\lim\limits_{z \to 1} \frac{\sqrt{z} - 1}{z - 1}$, ii) $\lim\limits_{z \to i} (\mathrm{Im} z - \mathrm{Re} z)$,
iii) $\lim\limits_{z \to 0} \frac{e^z + z\mathrm{Log} z}{1 - z^2 \mathrm{Arg} z}$, iv) $\lim\limits_{z \to 3} z^z$.

Exercice 6. Soit $f : \mathbb{C} \longrightarrow \mathbb{C}$ définie par :

$$f(z) = \begin{cases} \exp(\frac{-1}{z^4}) & \text{si} \quad z \neq 0, \\ 0 & \text{si} \quad z = 0. \end{cases}$$

a) Montrer que f vérifie les conditions de Cauchy-Riemann en $z_0 = 0$.
b) Montrer que f n'est pas $\mathbb{C}$-dérivable en $z_0 = 0$.

Safae

holomorphe $\Rightarrow$ $\mathbb{C}$-dérivable au voisinage de z_0 $\Leftrightarrow$ $\mathbb{C}$-dérivable en un petit disque

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Exercice 7. Les fonctions suivantes définies sur $\mathbb{C}$ sont-elles holomorphes ?

i) $f(z) = x^4y^5 + ixy^3$,
ii) $g(z) = \cos^2(x-y) + i\sin^2(x-y)$,
iii $h(z) = e^x \cos y - 2xy + i(e^x \sin y + x^2 - y^2)$.

Vérifier les conditions de Cauchy-Riemann.

Exercice 8. Soit $f : \mathbb{C} \longrightarrow \mathbb{C}$ une fonction telle que $f = u + iv$. Déterminer v pour que f soit holomorphe sur $\mathbb{C}$, dans les cas suivants :

1. $u(x,y) = 2xy$,
2. $u(x,y) = x^2 - y^2 + e^{-y}\sin x$.
3. $u(x,y) = 2x^3 - 6xy^2 + x^2 - y^2$.

Exercice 9. On définie les fonctions cos et sin sur $\mathbb{C}$ par :

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}.$$

1. Vérifier les identités suivantes :
   a) $\cos^2 z + \sin^2 z = 1$, $\quad \cos 2z = \cos^2 z - \sin^2 z$, pour tout $z \in \mathbb{C}$.
   b) $\cos z = \cos x \operatorname{ch} y - i \sin x \operatorname{sh} y$, $\quad \sin z = \sin x \operatorname{ch} y + i \cos x \operatorname{sh} y$, pour $z = x + iy$.
2. En déduire que les fonctions $\cos z$ et $\sin z$ ne sont pas bornées sur $\mathbb{C}$.

Exercice 10. Représenter les chemins suivants en précisant s'ils sont simples ou non. Calculer la longueur $l(\gamma)$.

1. $\gamma_1 : [0,1] \longrightarrow \mathbb{C}$, $\gamma_1(t) = t + it$.
2. $\gamma_2 : [-1,1] \longrightarrow \mathbb{C}$, $\gamma_2(t) = t^2 + it^2$.
3. $\gamma_3 : [0,4] \longrightarrow \mathbb{C}$, $\gamma_3(t) = t$ si $t \in [0,1]$, $\gamma_3(t) = 1 + i(t-1)$ si $t \in [1,2]$, $\gamma_3(t) = (3-t) + i$ si $t \in [2,3]$ et $\gamma_3(t) = i(4-t)$ si $t \in [3,4]$.

Exercice 11. Montrer que :

1. $\left|\displaystyle\int_\gamma \frac{dz}{z^3+1}\right| \leq \frac{2\pi}{21}$; où $\gamma(t) = 2e^{it}$ pour $t \in [-\frac{\pi}{6}, \frac{\pi}{6}]$.
2. $\left|\displaystyle\int_\gamma e^{iz^2}\right| \leq \frac{\pi}{4}\frac{1-e^{-R^2}}{R}$, où $\gamma(t) = Re^{it}$ pour $t \in [0, \frac{\pi}{4}]$. Indication : utiliser l'inégalité $\quad \sin t \geq \frac{2}{\pi}t$ pour $t \in [0, \frac{\pi}{2}]$.

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SMP 3

Analyse 3 : Série 2

Exercice 1. Soit $\gamma : [0,1] \longrightarrow \mathbb{C}$, définie par $\gamma(t) = e^{i\pi t}$ pour $t \in [0,1]$. Calculer $\displaystyle\int_\gamma \cos(nz)\, dz$, $n \in \mathbb{Z}$:

i) En utilisant les primitives,

ii) En utilisant le théorème de Cauchy.

Exercice 2. En utilisant la formule de Cauchy, calculer les intégrales suivantes :

a) $\displaystyle\int_{C(i,3)} \frac{\sin z}{z-i} dz$,

b) $\displaystyle\int_{C(0,4)} \frac{\cos z}{z^2 - i^2} dz$,

c) $\displaystyle\int_{C(0,2)} \frac{dz}{(z-1)^n (z-3)} dz$, $n \in \mathbb{N}$.

Exercice 3. Soit $I = \displaystyle\int_0^{+\infty} \cos(x^2) dx$ et $J = \displaystyle\int_0^{+\infty} \sin(x^2) dx$. Sachant que $K = \displaystyle\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, calculer les valeurs de I et J. Pour cela, considérer $f(z) = e^{iz^2}$ et le chemin $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, avec $\gamma_1(t) = t$ pour $t \in [0, R]$, $\gamma_2(\theta) = Re^{i\theta}$, $\theta \in [0, \frac{\pi}{4}]$ et $\gamma_3(t) = (R-t)e^{i\frac{\pi}{4}}$ pour $t \in [0, R]$.

Exercice 4. Soit $a \in \mathbb{R}^*$, $|a| \leq 1$. Montrer que :

$$\int_0^{+\infty} e^{-(1+ia)t^2} dt = \frac{\sqrt{\pi}}{2} \frac{1}{1+ia}.$$

Considérer $f(z) = e^{-z^2}$ et le triangle OAB, avec $O(0,0)$, $A(R,0)$ et $B(R, Ra)$ où $R > 0$.

Exercice 5. Déterminer les points singuliers isolés et leurs natures dans chacun des cas suivants :

a) $\dfrac{z^4}{(z^4+16)^2}$, b) $\dfrac{z^2 - \pi^2}{\sin^2 z}$,

c) $\dfrac{1 - \cos z}{\sin z}$, d) $\dfrac{1}{\cos(\frac{1}{z})}$.

Exercice 6. Calculer, à l'aide du théorème des résidus, l'intégrale :

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x} dx\,, \qquad a \in]0,1[.$$

Considérer le rectangle $[-R, R] \times [0, 2\pi]$, où $R > 0$.

Exercice 7. En utilisant le théorème des résidus, calculer les intégrales suivantes :

i) $\displaystyle\int_{\mathcal{C}(0,2)} \frac{dz}{z^4 + (1-i)z^3 - iz^2} dz.$

ii) $\displaystyle\int_{\mathcal{C}(0,1)} \frac{dz}{3z^2 + 10z + 3} dz.$

iii) $\displaystyle\int_{\mathcal{C}(0,1)} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$, $n \in \mathbb{N}^*$.

Exercice 8. Calculer les intégrales généralisées suivantes :

$$I = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+4)}\,, \qquad J = \int_0^{+\infty} \frac{x}{x^8+1} dx, \qquad K = \int_0^{+\infty} \frac{dx}{x^3+1}.$$

Exercice 9. Calculer les intégrales généralisées suivantes :

$$I = \int_0^{+\infty} \frac{x \sin x}{(x^2+1)^2} dx\,, \qquad J = \int_0^{+\infty} \frac{\sin^2 x}{x^2} dx.$$

Exercice 10. Calculer les intégrales définies suivantes :

i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{3\cos\theta + 5}$, ii) $\displaystyle\int_0^{2\pi} (\cos t)^{2n} dt$, $n \in \mathbb{N}$,

iii) $\displaystyle\int_{\mathcal{C}(0,1)} \frac{|dz|}{|z-a|^2}$, $a \in \mathbb{C}$, $|a| \neq 1$.

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Exercice 1 : Ecrire sous forme algébrique $z = x + iy$ les nombres.

i) $\operatorname{Log}(-e^2)$

ii) $\operatorname{Log}(1 - i\sqrt{3})$

iii) $(-1)^i$

iv) $i^{\operatorname{Log} i}$

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Rappel : La détermination principale du logarithme est définie par : $\operatorname{Log} z = \operatorname{Log}|z| + i \operatorname{Arg} z$, $z \in \mathbb{C}^*$

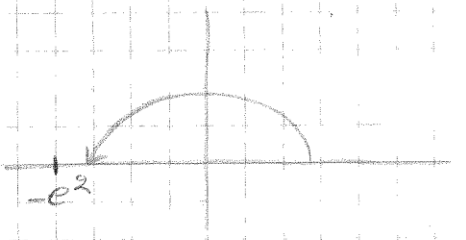
où $\operatorname{Arg} z$ est la détermination principale de l'argument, càd tq : $-\pi < \operatorname{Arg} z \leq \pi$

i) $z = -e^2$

$$\operatorname{Log}(-e^2) = \operatorname{Log}|-e^2| + i \operatorname{Arg}(-e^2)$$

$$|-e^2| = e^2, \quad \operatorname{Log}|-e^2| = \operatorname{Log} e^2 = 2 \operatorname{Log} e = 2.$$

$$\operatorname{Arg}(-e^2) = \pi$$



Donc $\operatorname{Log}(-e^2) = 2 + i\pi$

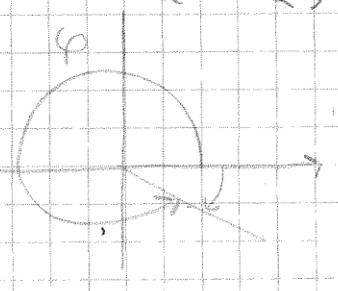
ii) $\operatorname{Log}(1 - i\sqrt{3})$

$$\operatorname{Log}(1 - i\sqrt{3}) = \operatorname{Log}|1 - i\sqrt{3}| + i \operatorname{Arg}(1 - i\sqrt{3})$$

$$|1 - i\sqrt{3}| = \sqrt{1 + 3} = 2$$

$\mathrm{Arg}(1-i\sqrt{3}) = -\frac{\pi}{3}$, $1-i\sqrt{3} = 2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$

$-\pi < \frac{3\pi}{2}+\frac{\pi}{2}-\frac{\pi}{3} = 2\pi-\frac{\pi}{3}+2k\pi \leq \pi$



$\varphi = 2\pi-\frac{\pi}{3}$, $-\pi < \varphi + 2k\pi \leq \pi$

$\Rightarrow \quad -\pi < 2\pi-\frac{\pi}{3}+2\pi k \leq \pi$

$-1 < 2(k+1) \leq 1+\frac{1}{3}$

$k+1 \leq \frac{1}{2}+\frac{1}{6} < 1 \Rightarrow k < 0$

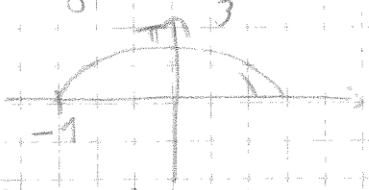
$-1 < -\frac{1}{3} = \frac{1}{6}-\frac{1}{2} \leq k+1 \quad \Rightarrow k \leq -1$

$\Rightarrow k > -2 \Rightarrow k \geq -1$

Donc $\quad \mathrm{Log}(1-i\sqrt{3}) = \mathrm{Log}|1-i\sqrt{3}| + i\,\mathrm{Arg}(1-i\sqrt{3})$

$\mathrm{Log}(1-i\sqrt{3}) = \mathrm{Log}\,2 - i\frac{\pi}{3}$

iii) $(-1)^i$



$z^\alpha = e^{\alpha\,\mathrm{Log}\,z}$

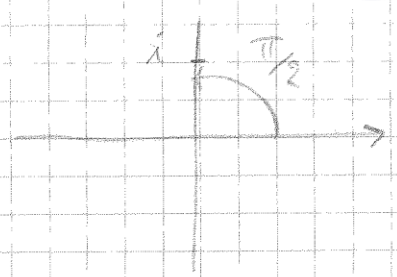
$(-1)^i = e^{i\,\mathrm{Log}(-1)}$ $\qquad \mathrm{Arg}(-1) = \pi$

$\mathrm{Log}(-1) = \mathrm{Log}|-1| + i\,\mathrm{Arg}(-1)$

$= \mathrm{Log}(1) + i\pi$

$= i\pi$

$(-1)^i = e^{i\,i\pi} = e^{-\pi}$



iv) $i^{\mathrm{Log}(i)}$

$i^{\mathrm{Log}(i)} = e^{\mathrm{Log}\,i\,\mathrm{Log}\,i}$

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$$\operatorname{Log} i = \operatorname{Log}|i| + i \operatorname{Arg}(i)$$
$$= \operatorname{Log} 1 + i\,\pi/2$$
$$\operatorname{Log} i = i\,\frac{\pi}{2}$$
$$i^{\operatorname{Log} i} = e^{\left(i\frac{\pi}{2}\right)^2} = e^{-\frac{\pi^2}{4}}$$

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Exercice 2 :

soient $z_1, z_2 \in \mathbb{C}^*$

i) Montrer que
$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$$

ii) Montrer que : $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$

ssi $-\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 \leq \pi$

iii) Mque si $|z| < 1$
$$\operatorname{Log}(1 - z^2) = \operatorname{Log}(1 - z) + \operatorname{Log}(1 + z)$$

i) soit $z_1, z_2 \in \mathbb{C}^*$

w_1 est un logarithme de z_1 si $e^{w_1} = z_1$
$$w_1 = \log z_1$$

w_2 est un logarithme de z_2 si $e^{w_2} = z_2$
$$w_2 = \log z_2$$

soit $w = w_1 + w_2 = e^{z}$

$$e^{w} = e^{w_1 + w_2} = e^{w_1} \times e^{w_2} = z_1 \times z_2$$

donc $w = w_1 + w_2$ est un logarithme de $z_1 z_2$.

$w_1 = \text{Log } z_1$, $w_2 = \text{Log } z_2$.

et $w = \text{Log } z_1 z_2$.

donc : $\log(z_1 \times z_2) = \log z_1 + \log z_2$

ii) supposons : $\text{Log}(z_1 z_2) = \text{Log } z_1 + \text{Log } z_2$

$\text{Log}(z_1 z_2) = \text{Log}|z_1 z_2| + i \,\text{Arg}(z_1 z_2)$.

avec $-\pi < \text{Arg}(z_1 z_2) \leqslant \pi$

$\text{Log } z_1 = \text{Log}|z_1| + i \,\text{Arg } z_1$

$\text{Log } z_2 = \text{Log}|z_2| + i \,\text{Arg } z_2$

$\text{Log } z_1 + \text{Log } z_2 = \text{Log}|z_1| + \text{Log}|z_2| + i(\text{Arg } z_1 + \text{Arg } z_2)$

$= \text{Log}|z_1 \times z_2| + i(\text{Arg } z_1 + \text{Arg } z_2)$

or :

$\text{Log}(z_1 z_2) = \text{Log}|z_1 z_2| + i \,\text{Arg}(z_1 z_2)$

$= \text{Log}|z_1 z_2| + i(\text{Arg } z_1 + \text{Arg } z_2)$

$\Rightarrow \text{Arg}(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2$

$\Rightarrow -\pi < \text{Arg } z_1 + \text{Arg } z_2 \leqslant \pi$

$\Leftarrow /$ supposons : $-\pi < \text{Arg } z_1 + \text{Arg } z_2 \leqslant \pi$

iii) on a : $\log(z_1 z_2) = \log z_1 + \log z_2$

or $\log(z_1 z_2) = \text{Log}(z_1 z_2) + 2k\pi i \quad k \in \mathbb{Z}$

$$\log z_1 = \operatorname{Log} z_1 + 2\ell\pi i \quad , \quad \ell \in \mathbb{Z}$$
$$\log z_2 = \operatorname{Log} z_2 + 2m\pi i , \quad m \in \mathbb{Z}$$

donc : $\operatorname{Log}(z_1 z_2) + 2k\pi i = \operatorname{Log} z_1 + 2\ell\pi i + \operatorname{Log} z_2 + 2m\pi i$

$$\Rightarrow \operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2\ell\pi i + 2m\pi i - 2k\pi i$$
$$= \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2p\pi i$$
$$p = \ell + m - k \in \mathbb{Z}$$

$$\Rightarrow \operatorname{Log}(z_1 z_2) = \operatorname{Log}|z_1 z_2| + i \operatorname{Arg}(z_1 z_2)$$

$$\operatorname{Log} z_1 = \operatorname{Log}|z_1| + i \operatorname{Arg} z_1$$

$$\operatorname{Log} z_2 = \operatorname{Log}|z_2| + i \operatorname{Arg} z_2$$

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donc :

$$\cancel{\operatorname{Log}|z_1 z_2|} + i \operatorname{Arg}(z_1 z_2) = \cancel{\operatorname{Log}|z_1|} + \cancel{\operatorname{Log}|z_2|} + i \operatorname{Arg} z_1 + i \operatorname{Arg} z_2 + 2p\pi i$$

$$\Rightarrow \cancel{i} \operatorname{Arg}(z_1 z_2) = \cancel{i} \operatorname{Arg} z_1 + \cancel{i} \operatorname{Arg} z_2 + 2\pi p \cancel{i}$$

$$\Rightarrow \operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2\pi p$$

$$\Rightarrow \operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg}(z_1 z_2) - 2\pi p$$

$$\Rightarrow -\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 \leq \pi$$

Donc : $-\pi < \operatorname{Arg}(z_1 z_2) - 2\pi p \leq \pi$

$\Rightarrow \quad -\pi + 2\pi p < \operatorname{Arg}(z_1 z_2) \leq \pi + 2\pi p$

Puisque : $\operatorname{Arg}(z_1 z_2) \leq \pi$, donc

$-\pi + 2\pi p < \pi$

$\Rightarrow \quad 2\pi p < 2\pi \Rightarrow p < 1$

$\Rightarrow \quad p \leq 0$ car $p \in \mathbb{Z}$

De même : $\operatorname{Arg}(z_1 z_2) > -\pi$, donc

$\pi + 2\pi p > -\pi \Rightarrow 2p\pi > -2\pi$

$\Rightarrow \quad p > -1 \Rightarrow p \geq 0 \quad (p \in \mathbb{Z})$

$\Rightarrow \quad p = 0$

D'où : $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$

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Par conséquent :

$\operatorname{Log}(z_1 z_2) = \operatorname{Log}|z_1 z_2| + i \operatorname{Arg}(z_1 z_2)$

$= \operatorname{Log}|z_1| + \operatorname{Log}|z_2| + i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2)$

$= \operatorname{Log}|z_1| + i \operatorname{Arg} z_1 + \operatorname{Log}|z_2| + i \operatorname{Arg} z_2$

$= \operatorname{Log} z_1 + \operatorname{Log} z_2$

$\boxed{\begin{array}{l} z_1 = -1 \\ z_2 = i \end{array}}$

Rq: En générale:

$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$ n'est pas vraie

iii) on a: $1 - z^2 = (1-z)(1+z)$

$= z_1 \times z_2$

avec: $z_1 = z_1 = 1 - z$

et $z_2 = 1 + z$

Posons:

$z = x + iy$

$z_1 = 1 - (x+iy)$

$= 1 - x - iy$

$= a + ib$

avec: $\begin{cases} a = 1 - x \\ b = -y \end{cases}$

$z_2 = 1 + z = 1 + (x+iy)$

$= 1 + x + iy$

$= c + id$

avec: $\begin{cases} c = 1 + x \\ d = y \end{cases}$

$\mathrm{Arg}\, z_1$?

$z_1 = a + ib$, $a = 1 - x$, $y = b$

on a: $|z| = \sqrt{x^2 + y^2} < 1$

et $|x| \leq \sqrt{x^2+y^2} < 1$

$\Rightarrow -1 < x < 1$

En particulier $a = 1 - x > 0$

donc $\mathrm{Arg}\, z_1 = \mathrm{Arctg}\, \frac{b}{a}$

$\mathrm{tg} : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\longrightarrow \mathbb{R}$ bijective

$\mathrm{Arctg} : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$\Rightarrow \mathrm{Arg}\, z_1 = \mathrm{Arctg}\, \frac{b}{a} \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$z_2 = c + id$, avec $\begin{cases} c = 1 + x \\ d = y \end{cases}$

$c = 1 + x > 0$, donc

$\mathrm{Arg}\, z_2 = \mathrm{Arctg}\, \frac{d}{c} \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$-\frac{\pi}{2} < \mathrm{Arg}\, z_1 < \frac{\pi}{2}$

$-\frac{\pi}{2} < \mathrm{Arctg}\, z_2 < \frac{\pi}{2}$

$-\frac{\pi}{2} < \mathrm{Arg}\, z_1 < \frac{\pi}{2}$

$-\frac{\pi}{2} < \mathrm{Arg}\, z_2 < \frac{\pi}{2}$

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$-\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 < \pi$

D'après ii) on a

$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$

$\Rightarrow$ Si $|x| < 1$

$\operatorname{Log}(1-z^2) = \operatorname{Log}(1+z) + \operatorname{Log}(1-z)$

En particulier, si $|z| < 1$

$\operatorname{Log}(1-z^2) = \operatorname{Log}(1-z) + \operatorname{Log}(1+z)$

Exercice 3 :

Déterminer l'ensemble des $z \in \mathbb{C}$ vérifiant :

i) $\frac{1}{2} < |z-i| < 1$

$D = \left\{ z \in \mathbb{C} / \frac{1}{2} < |z-i| < 1 \right\}$

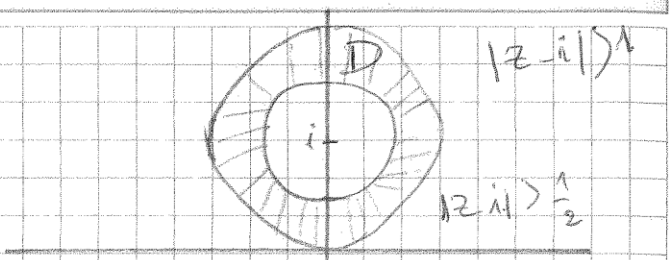
Posons :

$D_1 = \left\{ z \in \mathbb{C} / |z-i| < 1 \right\}$

$D_2 = \left\{ z \in \mathbb{C} / \frac{1}{2} < |z-i| \right\}$

$D = D_1 \cap D_2$

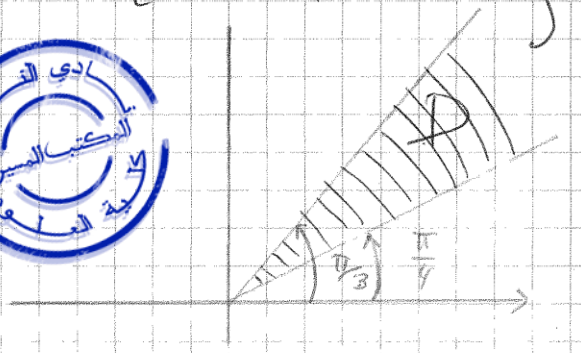
$D_1 = D(i, 1)$ disque ouvert de centre i de rayon 1



$D_2 = \mathbb{C} - \overline{D}(i, \frac{1}{2})$

D est une couronne de centre i de de rayons $\frac{1}{2}$ et 1

ii) $D = \left\{ z \in \mathbb{C} / \frac{\pi}{4} < \operatorname{Arg} z < \frac{\pi}{3} \right\}$

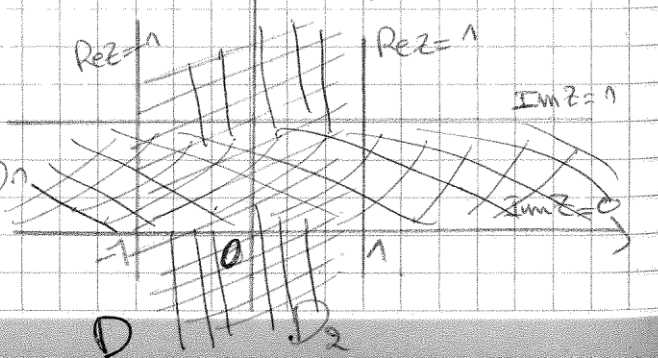


iii) $D = \left\{ z \in \mathbb{C} / 0 < \operatorname{Im} z < 1 \right.$ et $\left. -1 < \operatorname{Re} z < 1 \right\}$

$D_1 = \left\{ z \in \mathbb{C} / 0 < \operatorname{Im} z < 1 \right\}$

$D_2 = \left\{ z \in \mathbb{C} / -1 < \operatorname{Re} z < 1 \right\}$

$D = D_1 \cap D_2$



Exercice 4. $z \in \mathbb{C}^*$, $z = x+iy$

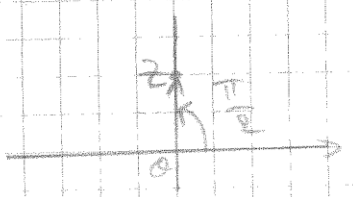
Mque.

$$\operatorname{Arg} z = \begin{cases} \text{i} & \operatorname{Arctg}\frac{y}{x} \quad x>0,\ y \in \mathbb{R} \\ \text{ii} & \pi + \operatorname{Arctg}\frac{y}{x} \quad \text{si } x<0,\ y \geq 0 \\ \text{iii} & -\pi + \operatorname{Arctg}\frac{y}{x} \quad \text{si } x<0,\ y<0 \\ \text{iv} & \frac{\pi}{2} \quad \text{si } x=0,\ y>0 \\ \text{v} & -\frac{\pi}{2} \quad \text{si } x=0,\ y<0 \end{cases}$$

iv) $x=0$, $y>0 \Rightarrow z = iy$

$$z = \overset{|z|}{\overset{\shortparallel}{y}}\, e^{i\frac{\pi}{2}}, \quad \frac{\pi}{2} \in]-\pi, \pi]$$

$$\operatorname{Arg} z = \frac{\pi}{2}$$



v) $x=0$, $y<0 \Rightarrow z = +iy$

$$z = (-y)(-i), \quad |z| = -y$$

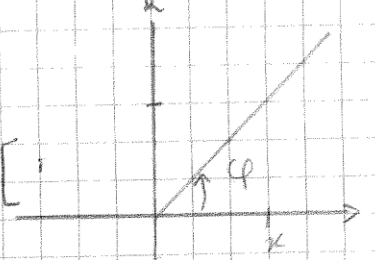
$$= (-y)e^{-i\frac{\pi}{2}}$$

$$= |z|\, e^{-i\frac{\pi}{2}}, \quad -\frac{\pi}{2} \in]-\pi, \pi]$$

donc : $\operatorname{Arg} z = -\frac{\pi}{2}$

i) $y>0$

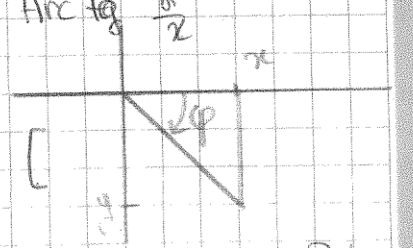
$\varphi \in]0, \frac{\pi}{2}[$



$\operatorname{tg}\varphi = \frac{y}{x}$, comme tg :

$]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$ bijective

Donc : $\varphi = \operatorname{Arctg}\frac{y}{x}$



ii) $y<0$

$\varphi \in]-\frac{\pi}{2}, 0[$

$\operatorname{tg}\varphi = \frac{y}{x}$, tg : $]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$ bij

$\operatorname{tg}\varphi = \frac{y}{x} \Rightarrow \varphi = \operatorname{Arctg}\frac{y}{x}$

$\operatorname{Arctg} : \mathbb{R} \longrightarrow]-\frac{\pi}{2}, \frac{\pi}{2}[$

ii) $x<0$, $y \geq 0$

i) $y=0$

$z = x$, $|z| = -x$

$= (-x)(-1)$

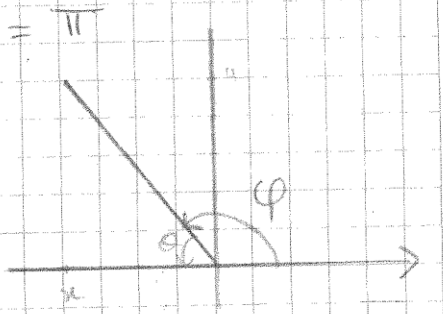
$= |z|\, e^{i\pi}$, $\pi \in]-\frac{\pi}{2}, \frac{\pi}{2}]$

, $\pi \in]-\pi, \pi]$

$\Rightarrow \operatorname{Arg} z = \pi$

ii) $y>0$

$\varphi = \operatorname{Arg} z$



cma : $\varphi \in]\frac{\pi}{2}, \pi[$

$\frac{\pi}{2} < \varphi < \pi$

$\Rightarrow \frac{\pi}{2} - \pi < \varphi - \pi < 0$

$\Rightarrow \varphi - \pi \in]-\frac{\pi}{2}, 0[$

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$\Rightarrow \varphi + \theta = \pi$

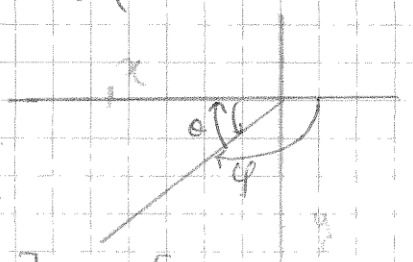
$\varphi - \pi = -\theta$

$$\left.\begin{array}{l} * \ tg(-\theta) = \dfrac{y}{x} \\ * \ -\theta = \varphi - \pi \in \left]-\frac{\pi}{2}, 0\right[\end{array}\right\} -\theta = \text{Arctg}\,\frac{y}{x}$$

$\Rightarrow \varphi - \pi = \text{Arctg}\,\dfrac{y}{x}$

$\Rightarrow \varphi = \pi + \text{Arctg}\,\dfrac{y}{x}$

iii) $x < 0$, $y < 0$



$\varphi \in \left]-\pi, -\frac{\pi}{2}\right[$

$\Rightarrow \varphi + \pi \in \left]0, \frac{\pi}{2}\right[$

$-\pi = \theta + \varphi \Rightarrow \varphi + \pi = -\theta \in \left]0, \frac{\pi}{2}\right[$

$$\left.\begin{array}{l} tg(-\theta) = \dfrac{y}{x} \\ -\theta \in \left]0, \frac{\pi}{2}\right[\end{array}\right\} \Rightarrow -\theta = \text{Arctg}\,\frac{y}{x}$$

$\Rightarrow \varphi + \pi = \text{Arctg}\,\dfrac{y}{x}$

$\Rightarrow \varphi = -\pi + \text{Arctg}\,\dfrac{y}{x}$

$z = x + iy = \sqrt{x^2+y^2} \times \left(\dfrac{x}{\sqrt{x^2+y^2}} + i\dfrac{y}{\sqrt{x^2+y^2}}\right)$

$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$ bij

$\text{Arc}\sin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Exercice 5 :

calculer les limites :

i) $\lim\limits_{z \to 1} \dfrac{\sqrt{z} - 1}{z - 1}$

ii) $\lim\limits_{z \to i} (\text{Im}\, z - \text{Re}\, z)$

iii) $\lim\limits_{z \to 0} \dfrac{e^z + z \log z}{1 - z^2 \text{Arg}\, z}$

iv) $\lim\limits_{z \to 3} z^z$

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i)

Posons $f(z) = \sqrt{z}$, $z \in \mathbb{C} -]-\infty, 0]$

f est dérivable sur D et

$f'(z) = \dfrac{1}{2\sqrt{z}} \quad \forall z \in D$

$$\lim_{z \to 1} \frac{\sqrt{z} - 1}{z - 1} = \lim_{z \to 1} \frac{f(z) - f(1)}{z - 1}$$

$$= f'(1)$$

$$= \frac{1}{2\sqrt{1}}$$

$$= \frac{1}{2} \quad \text{car } 1 \in D$$

On peut faire le calcul directement :

Soit $z \in D(1,r)$ $\quad 0<r<1$

$\sqrt{z} = e^{\operatorname{Log}\sqrt{z}}$

$= e^{\frac{1}{2}\operatorname{Log} z}$

$(\sqrt{z})^2 = \left(e^{\frac{1}{2}\operatorname{Log} z}\right)^2 = e^{\operatorname{Log} z}$

$= z$

$z-1 = (\sqrt{z})^2 - 1$

$= (\sqrt{z}-1)(\sqrt{z}+1)$

Pour $z \neq 1$, $z \in D(1,r)$

$\frac{\sqrt{z}-1}{z-1} = \frac{\cancel{\sqrt{z}-1}}{(\cancel{\sqrt{z}-1})(\sqrt{z}+1)}$

$= \frac{1}{\sqrt{z}+1}$

$\lim_{z\to 1} \frac{\sqrt{z}-1}{z-1} = \lim_{z\to 1} \frac{1}{\sqrt{z}+1}$

$= \frac{1}{2}$

2°/ $\lim_{z\to i} (\operatorname{Im} z - \operatorname{Re} z)$

Si $z = 0 + iy \longrightarrow i$
$y \longrightarrow 1$

$f(z) = \operatorname{Im} z - \operatorname{Re} z$

$f(0+iy) = y - 0 = y \longrightarrow 1$
$y \longrightarrow 1$

$z = x + i \longrightarrow i$
$x \longrightarrow 0$

$f(x+i) = 1 - x \longrightarrow 1$
$x \longrightarrow 0$

Mque : $\lim_{z\to i} f(z) = 1$

$\forall \varepsilon > 0 \ \exists \eta > 0 : |z-i| < \eta$

$\Rightarrow |f(z) - 1| < \varepsilon$

soit $z = x + iy$

$|f(z) - 1| = |y - x - 1|$

$|z - i| = |x + iy - i|$

$= |x + (y-1)i|$

$= \sqrt{x^2 + (y-1)^2}$

$|y - x - 1| = |y - 1 - x|$

$\leq |y-1| + |x|$

$|f(z) - 1| \leq |y-1| + |x|$

or : $|x|^2 \leq x^2 + (y-1)^2$

$\Rightarrow |x| \leq \sqrt{x^2 + (y-1)^2} = |z-i|$

et $|y-1|^2 \leq x^2 + (y-1)^2$

$\Rightarrow |y-1| \leq \sqrt{x^2 + (y-1)^2} = |z-i|$

$\Rightarrow |x| + |y-1| \leq 2|z-i|$

D'où $|f(z) - 1| \leq 2|z-i| < \varepsilon$

pour $\varepsilon > 0$ $\exists \eta = \frac{\varepsilon}{2} > 0$ tq

$|z - i| < \eta \Rightarrow |f(z) - 1| < \varepsilon$

$\lim_{z \to i} f(z) = 1$

iii $\lim_{z \to 0} \frac{e^z + z \operatorname{Log} z}{1 - z^2 \operatorname{Arg} z}$

*) $\lim_{z \to 0} e^z = e^0 = 1$

α) $\lim_{z \to 0} z \operatorname{Log} z =$

Pour $z \in \mathbb{C}^*$

$\operatorname{Log} z = \operatorname{Log}|z| + i \operatorname{Arg}(z)$

$z \operatorname{Log} z = z \operatorname{Log}|z| + iz \operatorname{Arg}(z)$

•) $\lim_{z \to 0} z \operatorname{Log}|z|$

$|z \operatorname{Log}|z|| = |z| |\operatorname{Log}|z||$

$= |\,|z| \operatorname{Log}|z|\,|$

Posons $r = |z|$, alors $z \xrightarrow{a.} 0$

$\Leftrightarrow r = |z| \xrightarrow[z \to 0]{} 0$

$||z| \operatorname{Log}|z|| = |r \operatorname{Log} r|$

$\lim_{z \to 0} |z \operatorname{Log}|z|| = \lim_{r \to 0^+} |r \operatorname{Log} r|$

$= 0$

$\Rightarrow \lim_{z \to 0} z \operatorname{Log}|z| = 0$

•) $\lim_{z \to 0} z \operatorname{Arg} z$

on a : $-\pi < \operatorname{Arg} < \pi$

donc $|\operatorname{Arg} z| < \pi$

$|z \operatorname{Arg} z| = |z||\operatorname{Arg} z| < |z| \pi \xrightarrow[z \to 0]{} 0$

$\Rightarrow \lim_{z \to 0} |z \operatorname{Arg} z| = 0$

$\Rightarrow \lim_{z \to 0} z \operatorname{Arg} z = 0$

$|z^2 \operatorname{Arg} z| = |z|^2 |\operatorname{Arg} z|$

$< \pi |z|^2 \xrightarrow[z \to 0]{} 0$

$\Rightarrow \lim_{z \to 0} |z^2 \operatorname{Arg} z| = 0$

$\Rightarrow \lim_{z \to 0} z^2 \operatorname{Arg} z = 0$

Finalement

$\lim_{z \to 0} \frac{e^z + z \operatorname{Log} z}{1 - z^2 \operatorname{Arg} z} = 1$

α) $\lim_{z \to 3} z^z$

soit $r \in]0, 3[$

on a $D(3, r) \subset \mathbb{C} -]-\infty, 0] = D$

Posons $f(z) = z^z = e^{z \operatorname{Log} z}$

La fct Log est continue sur D.

$\Rightarrow$ Log est continue sur $D(3,r)$

$$\Rightarrow \lim_{z\to 3} f(z) = \lim_{z\to 3} e^{z\,\mathrm{Log}\,z} = e^{3\,\mathrm{Log}\,3}$$

$$= 3^3 = 27$$

Ex 6 :

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

$$f(z) = \begin{cases} \exp\left(\dfrac{-1}{z^4}\right), & \text{si } z \neq 0 \\ 0 & \text{si } z = 0 \end{cases}$$

a)

$f = u + iv$

Pour $z \neq 0$, posons $z = x + iy$

$$f(z) = \exp\left(\frac{-1}{z^4}\right)$$

$$\frac{1}{z^4} = \frac{\bar{z}^4}{z^4 \cdot \bar{z}^4} = \frac{\bar{z}^4}{(z\cdot\bar{z})^4} = \frac{\bar{z}^4}{(|z|^2)^4}$$

$$|z|^2 = x^2 + y^2$$

$$\bar{z}^4 = (x - iy)^4$$

$$= x^4 + 4x^3(-iy) + 6x^2(-iy)^2 + 4x(-iy)^3 + (-iy)^4$$

$$\bar{z}^4 = x^4 - i\,4x^3y - 6x^2y^2 + i\,4xy^3 + y^4$$

$$= x^4 - 6x^2y^2 + y^4 + i(4xy^3 - 4yx^3)$$

$$\frac{-\bar{z}^4}{(|z|^2)^4} = -\frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^4} - i\,\frac{4xy^3 - 4yx^3}{(x^2+y^2)^4}$$

$$\frac{-\bar{z}^4}{(|z|^2)^4} = g(x,y) + i\,h(x,y)$$

$$f(z) = \exp\big(g(x,y) + i\,h(x,y)\big)$$

$$= \exp g(x,y) \cdot e^{i\,h(x,y)}$$

$$= \exp g(x,y)\big(\cos h(x,y) + i \sin h(x,y)\big)$$

$$= u(x,y) + i\,v(x,y)$$

$$u(x,y) = \exp g(x,y) \cos h(x,y)$$

$$v(x,y) = \exp g(x,y) \sin h(x,y)$$

$$f(0) = 0 = u(0,0) + i\,v(0,0)$$

$$\Rightarrow u(0,0) = v(0,0)$$

les conditions de Cauchy Riemann en $z_0 = 0$

$$\frac{\partial u}{\partial x}(0,0) = \frac{\partial v}{\partial y}(0,0) \ ?$$

$$\frac{\partial u}{\partial y}(0,0) = -\frac{\partial v}{\partial x}(0,0) \ ?$$

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x - 0}$$

$$= \lim_{x\to 0} \frac{\exp g(x,0) \cos h(x,0)}{x}$$

$$g(x,0) = \frac{-x^4}{x^8}$$

$$= \frac{-1}{x^4} \qquad x \neq 0$$

$h(x,0) = 0$

$u(x,0) = \exp\left(\frac{-1}{x^4}\right)$ si $x \neq 0$

$$\lim_{x\to 0} \frac{u(x,0)}{x} = \lim_{x\to 0} \frac{\exp\left(\frac{-1}{x^4}\right)}{x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}(0,0) = 0$$

$$\frac{\partial v}{\partial y}(0,0) = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y - 0} = 0$$

$$v(0,y) = \exp(g(0,y)) \sin h(0,y)$$
$$= 0$$

$$\frac{\partial u}{\partial y}(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{\exp\left(\frac{-1}{x^4}\right)}{y} = 0$$

$$\frac{\partial v}{\partial x}(0,0) = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x - 0}$$

$$= 0$$

Finalement :

$$\frac{\partial u}{\partial x}(0,0) = 0 = \frac{\partial v}{\partial y}(0,0)$$

$$\frac{\partial u}{\partial y}(0,0) = 0 = -\frac{\partial v}{\partial x}(0,0)$$

Donc f vérifie les conditions de Cauchy-Riemann en $z_0 = 0$

b) Mque :

f n'est pas $\mathbb{C}$-dérivable en $z_0 = 0$.

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$$

$$z = 0 + iy \xrightarrow[y\to 0]{} 0$$

soit y, $f(0 + iy) = \exp\left(\frac{-1}{(iy)^4}\right)$

$$= \exp\left(\frac{-1}{y^4}\right)$$

si $z = re^{i\frac{\pi}{4}} \xrightarrow[r\to 0]{} 0$

$$f(z) = f\left(re^{i\frac{\pi}{4}}\right)$$

$$= \exp\left(\frac{-1}{r^4 e^{i\pi}}\right)$$

$$= \exp\left(\frac{1}{r^4}\right)$$

$$\lim_{r\to 0} \frac{f\left(re^{i\frac{\pi}{4}}\right)}{re^{i\pi/4}}$$

$$= \lim_{r\to 0} \frac{1}{e^{i\frac{\pi}{4}}} \frac{\exp\left(\frac{1}{r^4}\right)}{r} = \infty$$

donc f n'est pas $\mathbb{C}$-dérivable en $z_0 = 0$

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Exercice 7 :

i) $f(z) = x^4y^5 + i\,xy^3$

$f(z) = u(x,y) + i\,(x,y)$

avec : $\begin{cases} u(x,y) = x^4y^5 \\ v(x,y) = xy^3 \end{cases}$

$u : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$v : \mathbb{R}^2 \longrightarrow \mathbb{R}$

Les fonctions u et v sont de classe C^1 sur $\mathbb{R}^2$.

Condition des cauchy-Riemann :

$\frac{\partial u}{\partial x}(x,y) = 4x^3y^5$

$\frac{\partial v}{\partial y}(x,y) = 3xy^2$

$\frac{\partial u}{\partial y}(x,y) = 5x^4y^4$

$\frac{\partial v}{\partial x}(x,y) = y^3$

$\begin{cases} \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} 4x^3y^5 = 3xy^2 \\ 5x^4y^4 = -y^3 \end{cases}$

$\Leftrightarrow \begin{cases} 4x^3y^5 - 3xy^2 = 0 & (i) \\ 5x^4y^4 + y^3 = 0 & (ii) \end{cases}$

$\overset{(*)}{\Leftrightarrow} \begin{cases} xy^2(4x^2y^3 - 3) = 0 & (i) \\ y^3(5x^4y + 1) = 0 & (ii) \end{cases}$

(ii) $\Rightarrow$ $y^3 = 0$ ou $5x^4y + 1 = 0$

$\Rightarrow y = 0$ ou $5x^4y + 1 = 0$

Si $y = 0$, $x \in \mathbb{R}$, on a

$\frac{\partial u}{\partial x}(x,0) = 0 = \frac{\partial v}{\partial y}(x,0)$

$\frac{\partial u}{\partial y}(x,0) = 0 = -\frac{\partial v}{\partial x}(x,0)$

$y \neq 0$ (ii) $\Rightarrow 5x^4y + 1 = 0$

$\Rightarrow 5x^4y = -1$

Si $x = 0$ $5.0^4y + 1 \neq 0$

Si $x \neq 0$, (ii) $\Rightarrow y = \frac{-1}{x^4 5} \neq 0$

(i): $xy^2(4x^2y^3-3)=0$

$\Rightarrow 4x^2y^3=3$

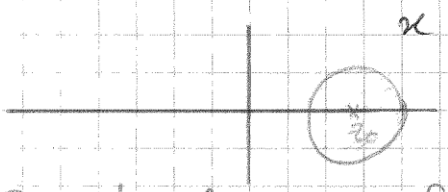
$\Rightarrow y^3=\dfrac{3}{4x^2}$

$\Rightarrow y=\sqrt[3]{\dfrac{3}{4x^2}}>0$

Conclusion:

les equations (*) sont vérifiées pour $z=x+i0$

$x\in\mathbb{R}$



Rappel: f est dite holomorphe en z_0 si f est $\mathbb{C}$-derivable au voisinage de z_0, c.à.d $\exists\, r>0$ tq f soit $\mathbb{C}$-derivable en tout pts de $D(z_0,r)$

L'ensemble où les equations de Cauchy-Riemann sont vérifiées est $A=\{z\in\mathbb{C}/\ \mathrm{Im}\,z=0\}$

comme $\forall z_0\in A$, $\nexists$ aucun rayon $r>0$ tq $D(z_0,r)\subset A$, f n'est pas holomorphe en tout point de $\mathbb{C}$

ii) $g(z)=\cos^2(x-y)+i\sin^2(x-y)$

$u(x,y)=\cos^2(x-y)$

$v(x,y)=\sin^2(x-y)$

$u,v:\ \mathbb{R}^2\longrightarrow\mathbb{R}$ sont de classe C^1 sur $\mathbb{R}^2$.

$\dfrac{\partial u}{\partial x}(x,y)=2\cos(x-y)(-\sin(x-y))$

$=-2\cos(x-y)\sin(x-y)$

$\dfrac{\partial v}{\partial y}(x,y)=-2\sin(x-y)\cos(x-y)$

$\dfrac{\partial u}{\partial y}(x,y)=2\cos(x-y)\sin(x-y)$

$\dfrac{\partial v}{\partial x}(x,y)=2\sin(x-y)\cos(x-y)$

donc:

$\dfrac{\partial u}{\partial x}(x,y)=\dfrac{\partial v}{\partial y}(x,y)$

$\forall (x,y)\in\mathbb{R}^2$

$\dfrac{\partial u}{\partial y}(x,y)=-\dfrac{\partial v}{\partial x}(x,y)$

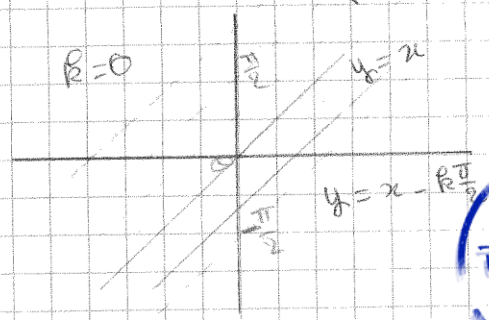
$\Leftrightarrow 2\cos(x-y)\sin(x-y)=2\sin(x-y)\cos(x-y)$

$\Leftrightarrow 2\,2\cos(x-y)\sin(x-y)=0$

$\Leftrightarrow \sin(2x-2y)=0$

$\Leftrightarrow 2x-2y = k\pi$ où $k\in\mathbb{Z}$

$\Leftrightarrow y = x - k\frac{\pi}{2}, \ k\in\mathbb{Z}$



La fonction g n'est pas holomorphe.

iii) $h(z) = e^x\cos y - 2xy + i(e^x\sin y + x^2 - y^2)$

$u(x,y) = e^x\cos y - 2xy$

$v(x,y) = e^x\sin y + x^2 - y^2$.

$u,v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ de classe C^1 sur $\mathbb{R}^2$.

$\frac{\partial u}{\partial x}(x,y) = e^x\cos y - 2y$

$\frac{\partial v}{\partial y}(x,y) = e^x\cos y - 2y$

$\frac{\partial u}{\partial y}(x,y) = -e^x\sin y - 2x$

$\frac{\partial v}{\partial x}(x,y) = e^x\sin y + 2x$

donc :

$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$

$= e^x\cos y - 2y$

$\forall (x,y)\in\mathbb{R}^2$

$\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$

$= -(e^x\sin y + 2x)$

$\forall (x,y)\in\mathbb{R}^2$.

<u>Conclusion</u> :

$u,v : \mathbb{R}^2 \longrightarrow \mathbb{R}$

sont de classe C^1 sur $\mathbb{R}^2$,

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \end{cases}$$

$\forall (x,y)\in\mathbb{R}^2 \simeq \mathbb{C}$

Donc $h = u + iv$ est holomorphe sur $\mathbb{C}$ car h est $\mathbb{C}$-dérivable en tout point de $\mathbb{C}$.

$h(z) = e^x\cos y + ie^x\sin y - 2xy + i(x^2-y^2)$

$= e^x(\cos y + i\sin y) + i(x^2-y^2+i2xy)$

$h(z) = e^{x} e^{iy} + i(x+iy)^2$

$= e^{x+iy} + iz^2$

$h(z) = e^{z} + iz^2 \quad \forall z \in \mathbb{C}$

h est holomorphe sur $\mathbb{C}$.

Exercice 8 :

$f : \mathbb{C} \longrightarrow \mathbb{C}$, $f = u + iv$

Trouver v pour que f soit holomorphe sur $\mathbb{C}$!

1) $u(x,y) = 2xy$

f est holomorphe sur $\mathbb{C}$

$\Rightarrow f$ vérifiée les conditions de Cauchy-Riemann.

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 2y \quad (i)$$

$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) = -2x \quad (ii)$$

Fixons $x \in \mathbb{R}$, $(i) \Rightarrow$

$$\frac{\partial v}{\partial y}(x,y) = 2y$$

$$\Rightarrow v(x,y) = \frac{2}{2}y^2 + c(x)$$

$$= y^2 + c(x)$$

$c(x) = v(x,y) - y^2$ est derivable

$$\frac{\partial v}{\partial x}(x,y) = 0 + c'(x) = -2x$$

$\Rightarrow c'(x) = -2x$

$\Rightarrow c(x) = -x^2 + d$, avec $d \in \mathbb{R}$

Donc :

$v(x,y) = y^2 - x^2 + d$, $d \in \mathbb{R}$

Finalement :

$f(z) = 2xy + i(y^2 - x^2 + d)$

$= 2xy - i(x^2 - y^2) + id$

$f(z) = -i\left[x^2 - y^2 + 2ixy\right] + id$

$= -i(x+iy)^2 + e$, $e \in \mathbb{C}$

$= -iz^2 + e$, $e \in \mathbb{C}$

2) $u(x,y) = x^2 - y^2 + e^{-y} \sin x$

f est holomorphe sur $\mathbb{C}$

$\Rightarrow f$ vérifie les conditions de Cauchy Riemann.

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 2x + e^{-y}\cos x \quad (i)$$

$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) = -\left(-2y - e^{-y}\sin x\right) \quad (ii)$$

Fixons $x \in \mathbb{R} \Rightarrow (i) \Rightarrow$

$$\frac{\partial v}{\partial y}(x,y) = 2x + e^{-y}\cos x$$

$$v(x,y) = 2xy + (-e^{-y}\cos x) + c(x)$$

$$c(x) = v(x,y) - 2xy + e^{-y}\cos x$$

$c(x)$ est derivable.

Fixons y :

$(ii) \Rightarrow$

$$\frac{\partial v}{\partial x}(x,y) = \cancel{2y} + e^{-y}\cancel{\sin x} + c'(x)$$

$$= \cancel{2y} + e^{-y}\cancel{\sin}(x)$$

$$\Rightarrow c'(x) = 0$$

$$\Rightarrow c(x) = d, \ d \in \mathbb{R}$$

Donc $v(x,y) = 2xy - e^{-y}\cos x + d, \ d \in \mathbb{R}$

Finalement :

$$f(z) = x^2 - y^2 + e^{-y}\sin x + i(2xy - e^{-y}\cos x + d)$$

$$f(z) = x^2 - y^2 + i2xy + e^{-y}\sin x - ie^{-y}\cos x + di$$

$$= (x+iy)^2 - ie^{-y}(\cos x + i\sin x) + id$$

$$= z^2 - ie^{-y}e^{ix} + e, \ e \in \mathbb{R}$$

$$= z^2 - ie^{i(x+iy)} + e \in \mathbb{C}$$

$$f(z) = z^2 - ie^{iz} + e \in \mathbb{C}$$

3) $u(x,y) = 2x^3 - 6xy^2 + x^2 - y^2$

Conditions de Cauchy-Riemann :

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$$

$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y)$$

u est de classe C^1 sur $\mathbb{R}^2$.

$$\frac{\partial u}{\partial x}(x,y) = 6x^2 - 6y^2 + 2x$$

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$$

$$= 6x^2 - 6y^2 + 2x$$

Fixons x, donc

$$v(x,y) = 6x^2y - \frac{6}{3}y^3 + 2xy + c(x)$$

$$= 6x^2y - 2y^3 + 2xy + c(x)$$

$$\frac{\partial u}{\partial y}(x,y) = -12xy - 2y$$

$$\frac{\partial v}{\partial x}(x,y) = 12xy + 2y + C'(x)$$

$$= -(-12xy - 2y)$$

$$\frac{\partial v}{\partial x}(x,y) = 12xy + 2y$$

$$\Rightarrow C'(x) = 0$$

$$\Rightarrow C(x) = d, \text{ avec } d \in \mathbb{R}$$

D'où

$$v(x,y) = 6x^2y - 2y^3 + 2xy + d \quad d \in \mathbb{R}.$$

$$* \; f(z) = 2x^3 - 6xy^2 + x^2 - y^2 + i(6x^2y - 2y^3 + 2xy) + id.$$

$$= (x^2 - y^2 + i2xy) + 2[x^3 - 3xy^2 + i(3x^2y - y^3)] + id$$

$$= z^2 + 2z^3 + id, \; d \in \mathbb{R}.$$

Exercice 9 :

On définit :

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \; z \in \mathbb{C}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \; z \in \mathbb{C}$$

1°/ a-

$$\cos^2 z = \left(\left(\frac{e^{iz} + e^{-iz}}{2}\right)\right)^2$$

$$= \frac{1}{4}\left[e^{2iz} + e^{-i2z} + 2\right]$$

$$\sin^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= -\left(\frac{e^{2iz} + e^{-2iz} + 2}{4}\right)$$

$$= -\frac{1}{4}\left(e^{2iz} + e^{-2iz} + (-2)\right)$$

$$\cos^2 z + \sin^2 z = \cancel{\frac{1}{4}e^{2iz}} + \cancel{\frac{1}{4}e^{-i2z}} + \frac{1}{2} - \cancel{\frac{1}{4}e^{2iz}} - \cancel{\frac{1}{4}e^{-2iz}} + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$\cos^2 z + \sin^2 z = 1$$

donc : $\cos^2 z + \sin^2 z = 1 \quad \forall z \in \mathbb{C}$

a)

$$\cos^2 z - \sin^2 z = \frac{1}{4}\left(e^{2iz} + e^{2iz} + \cancel{2} + e^{2iz} + e^{-2iz} - \cancel{2}\right)$$

$$\cos^2 z - \sin^2 z = \frac{1}{\cancel{4}_{2}}\left(\cancel{2}e^{2iz} + \cancel{2}e^{-2iz}\right)$$

$$\cos^2 z - \sin^2 z = \frac{e^{2iz} + e^{-2iz}}{2} = \cos(2z)$$

$$\cos 2z = \cos^2 z - \sin^2 z$$
$$= 2\cos^2 z - 1$$
$$\cos 2z = 1 - 2\sin^2 z$$

c) $\sin 2z = 2\sin z \cos z \quad \forall z \in \mathbb{C}$

b) pour $z = x + iy \in \mathbb{C}$

$$\cos z = \cos x \, \mathrm{ch}\, y - i \sin x \, \mathrm{sh}\, y.$$

Rappel:

$$\mathrm{ch}\, y = \frac{e^{y} + e^{-y}}{2} \qquad y \in \mathbb{R}.$$
$$\mathrm{sh}\, y = \frac{e^{y} - e^{-y}}{2}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$= \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right)$$
$$= \frac{1}{2}\left(e^{ix+(-y)} + e^{-ix+y}\right)$$
$$= \frac{1}{2}\left(e^{-y} \times e^{ix} + e^{y} \times e^{-ix}\right)$$
$$\cos z = \frac{1}{2}\left(e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)\right)$$
$$= \frac{1}{2}\left[\cos x\left(e^{-y} + e^{y}\right) - i \sin x\left(e^{y} - e^{-y}\right)\right]$$
$$\cos z = \cos x \, \frac{e^{y} + e^{-y}}{2} - i \sin x \, \frac{e^{y} - e^{-y}}{2}$$
$$= \cos x \, \mathrm{ch}\, y - i \sin x \, \mathrm{sh}\, y$$

$$\cos z = \cos x \, \mathrm{ch}\, y - i \sin x \, \mathrm{sh}\, y$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$= \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right)$$
$$\sin z = \frac{1}{2i}\left(e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\right)$$
$$\sin z = \frac{1}{2i}\left[\cos x\left(e^{-y} - e^{y}\right) + i \sin x\left(e^{y} + e^{-y}\right)\right]$$
$$\sin z = \cos x \, \frac{e^{-y} - e^{y}}{2i} + \sin x \, \frac{e^{y} + e^{-y}}{2}$$
$$\sin z = \sin x \, \mathrm{ch}\, y - i \cos \mathrm{sh}\, y$$
$$\sin z = \sin x \, \mathrm{ch}\, y - i \cos \mathrm{sh}\, y.$$

On définit aussi

$$\mathrm{ch}\, z = \frac{e^{z} + e^{-iz}}{2} \qquad z \in \mathbb{C}$$
$$\mathrm{sh}\, z = \frac{e^{z} - e^{-iz}}{2}$$

On obtient:

$$\cos z = \mathrm{ch}\, iz \qquad z \in \mathbb{C}$$
$$\sin z = \frac{1}{i} \mathrm{sh}\, iz$$

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2°/ $\cos z$ et $\sin z$ ne sont pas bornées sur $\mathbb{C}$:

$$|\cos z|^2 = (\cos x \operatorname{ch} y)^2 + (\sin x \operatorname{sh} y)^2$$

$$= \cos^2 x \operatorname{ch}^2 y + \sin^2 x \operatorname{sh}^2 y$$

On a : $\operatorname{ch}^2 y - \operatorname{sh}^2 y = 1 \quad \forall y \in \mathbb{R} \Rightarrow \operatorname{ch}^2 y = 1 + \operatorname{sh}^2 y$

$$\Rightarrow |\cos z|^2 = \cos^2 x (1 + \operatorname{sh}^2 y) + \sin^2 x \operatorname{sh}^2 y$$

$$= \cos^2 x + (\cos^2 x + \sin^2 x) \operatorname{sh}^2 y$$

$$= \cos^2 x + \operatorname{sh}^2 y \geqslant \operatorname{sh}^2 y$$

$|\cos z|^2 \geqslant \operatorname{sh}^2 y \Rightarrow |\cos z| \geqslant |\operatorname{sh} y|$

comme $\lim\limits_{y \to +\infty} |\operatorname{sh} y| = +\infty$ et $\lim\limits_{y \to -\infty} |\operatorname{sh} y| = +\infty$

$\Rightarrow |\cos z|$ ne peut pas être majorée ; car si $\exists M > 0$

tq $|\cos z| \leqslant M \quad \forall z \in \mathbb{C}$,

$\Rightarrow |\operatorname{sh} y| \leqslant M \quad \forall y \in \mathbb{R}$ ce qui est impossible.

Conclusion :

$\cos z$ n'est pas bornée dans $\mathbb{C}$

**) $|\sin z|^2 = (\sin x \operatorname{ch} y)^2 + (\cos x \operatorname{sh} y)^2$

$$= \sin^2 x \operatorname{ch}^2 y + \cos^2 x \operatorname{sh}^2 y$$

$$= \sin^2 x (1 + \operatorname{sh}^2 y) + \cos^2 x \operatorname{sh}^2 y$$

$$= \sin^2 x + \operatorname{sh}^2 y (\sin^2 x + \cos^2 x)$$

$$= \sin^2 x + \operatorname{sh}^2 y \geqslant \operatorname{sh}^2 y$$

$\Rightarrow |\sin z|^2 \geqslant \operatorname{sh}^2 y$

$\Rightarrow \sin^2 z \geqslant |\operatorname{sh} y|$

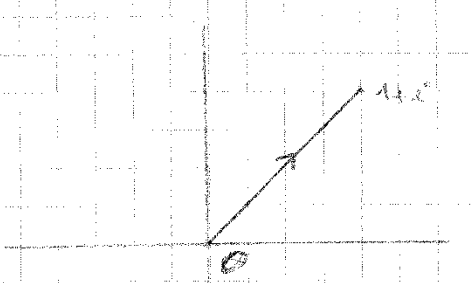
puisque $y \longrightarrow |\operatorname{sh} y|$ n'est pas majorée, alors $\sin z$ ne peut pas être bornée dans $\mathbb{C}$.

$$|\cos z|^2 + |\sin z|^2 = 1 + 2\operatorname{sh}^2 y$$

Exercice 10 :

1°/ $\gamma_1 : [0,1] \longrightarrow \mathbb{C}$

$t \longrightarrow t + it$



$\gamma_1(t) = x(t) + i\,y(t) \qquad t \in [0,1]$

Point de départ $\gamma_1(0) = 0$

Point d'arrivé $\gamma_1(1) = 1+i$

$x(t) = t = y(t) \qquad \forall t \in [0,1]$.

$\gamma_1(t) = \gamma_1(s) \Leftrightarrow t + it = s + is$

$\Leftrightarrow (1+i)t = (1+i)s \Leftrightarrow t = s$ donc $\forall\, t \neq s \quad \gamma_1(t) \neq \gamma_1(s)$

donc γ_1 est un chemin simple.

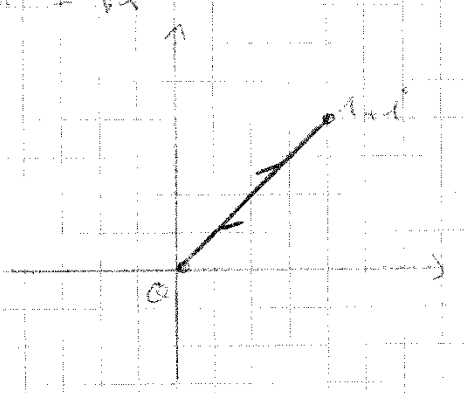
γ est un chemin de classe C^1 car $x : t \longrightarrow t$ et $y : t \longrightarrow t$ sont de classe C^1.

$\ell(\gamma_1) = \int_0^1 |\gamma_1'(t)|\,dt \quad , \quad \gamma_1'(t) = x'(t) + i\,y'(t) = 1 + i$

$\ell(\gamma_1) = \int_0^1 |1+i|\,dt = \int_0^1 \sqrt{2}\,dt = \sqrt{2}$

2°/ $\gamma_2 : [-1,1] \longrightarrow \mathbb{C}$

$t \longrightarrow t^2 + it^3$



Point de départ

$\gamma_2(-1) = 1 + i$

Point d'arrivé $\gamma_2(1) = 1 + i$

$\gamma_2(t) = x(t) + i y(t) \qquad t \in [-1,1]$

γ_2 est continue sur $[-1,1]$.

$\gamma_2(-1) = 1+i = \gamma_2(1)$

donc γ_2 est un chemin fermé $\quad x(t) = t^2 = y(t) \quad \forall t \in [-1,1]$

le chemin γ_2 n'est pas simple car $\gamma_2(-t) = \gamma_2(t)$

$\forall t \in [-1,1]$.

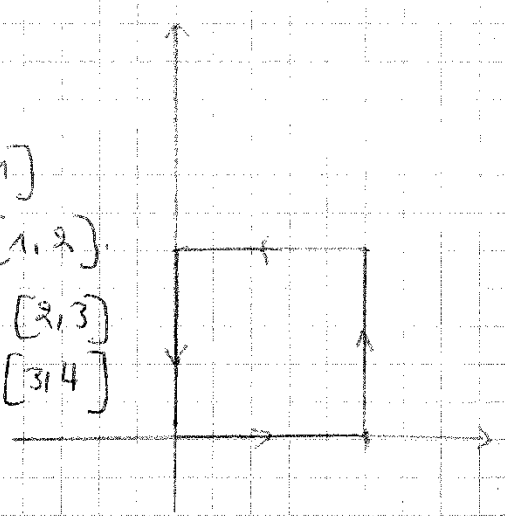
$$\ell(\gamma_2) = \int_{-1}^{1} |\gamma'(t)|\, dt = \int_{-1}^{1} |2t + 2ti|\, dt$$

$$= \int_{-1}^{1} 2|t|\,|1+i|\, dt = \sqrt{2}\, 4 \int_{0}^{1} t\, dt$$

$$= \sqrt{2} \cdot 4 \left[\frac{t^2}{2}\right]_0^1 = 2\sqrt{2}$$

3°/ $\gamma_3 : [0,4] \longrightarrow \mathbb{C}$

$$\gamma_3(t) = \begin{cases} t & t \in [0,1] \\ 1 + i(t-1) & t \in [1,2] \\ 3 - t + i & t \in [2,3] \\ i(4-t) & t \in [3,4] \end{cases}$$



$\gamma_3(t) = x(t) + i y(t)$

$x(t) = t \;,\; y(t) = 0 \qquad t \in [0,1]$

$x(t) = 1 \;,\; y(t) = (t-1) \qquad t \in [1,2]$

$x(t) = 3 - t \;,\; y(t) = 1 \qquad t \in [2,3]$

$x(t) = 0 \;,\; y(t) = 4 - t \qquad t \in [3,4]$

γ_3 est un chemin de classe C^1 par morceaux

$\gamma_3(0) = 0 = \gamma_3(4)$.

γ_3 est un chemin fermé, γ_3 est simple car γ_3 ~~est simple~~

$\gamma_3(s) \neq \gamma_3(t) \quad \forall t \neq s \in]0,4[$.

$$l(\gamma_3) = \int_0^1 |\gamma_3'(t)|dt + \int_1^2 |\gamma_3'(t)|dt + \int_2^3 |\gamma_3'(t)|dt + \int_3^4 |\gamma_3'(t)|dt$$

$$= 4$$

$$l(\gamma_3) = \int_0^1 dt + \int_1^2 |i| + \int_2^3 |-1|dt + \int_3^4 |-i|dt.$$

$$= 1 + (2-1) + (3-2) + (4-3)$$

$$= 4$$

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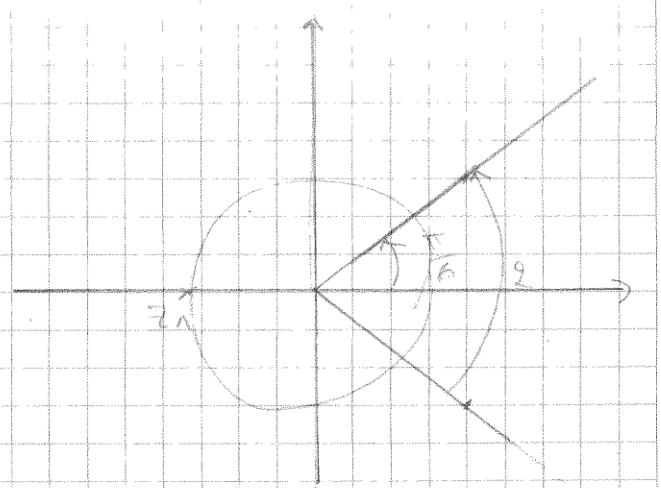
Exercice 11 :

1°/ $\gamma : \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \longrightarrow \mathbb{C}$

$t \longrightarrow 2e^{it}$

$$\left|\int_\gamma \frac{dz}{z^3+1}\right| \leq \frac{2\pi}{21}$$



soit $f(z) = \frac{1}{z^3+1}$

$D_f = \{z \in \mathbb{C} \,/\, z^3+1 \neq 0\}$

$z^3+1 = 0 \iff z^3 = -1 = e^{i\pi}$

$\Rightarrow z_k = e^{i\frac{\pi}{3} + 2k\frac{\pi}{3}i} \quad k = 0, 1, 2$

$z_0 = e^{i\frac{\pi}{3}}, \quad z_1 = e^{i\pi} = -1, \quad z_2 = e^{i\frac{\pi}{3} + \frac{4}{3}i}$

soit $\Omega = D_f = \mathbb{C} - \{z_0, z_1, z_2\}$

$$f : \Omega \longrightarrow \mathbb{C}$$
$$z \longrightarrow \frac{1}{z^3+1} \quad \text{est continue}$$

si $t \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ $\gamma^3(t) = \left(2e^{it}\right)^3 \neq -1$

donc $\gamma : \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \longrightarrow \Omega$

est un chemin de classe C^1 continue dans Ω donc.

$$\int_\gamma f(z)\,dz = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} f\circ\gamma(t)\,\gamma'(t)\,dt.$$

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} f\left(2e^{it}\right).2ie^{it}\,dt\right| \leqslant \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left|f(2e^{it})\right| \times \left|2ie^{it}\right| dt$$
$$\leqslant \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left|f(2e^{it})\right| 2\,dt$$

Rappel :

si $a, b \in \mathbb{C}$, on a :

$$\big||a| - |b|\big| \leqslant |a+b| \leqslant |a| + |b|$$

$$\big||a| - |b|\big| \leqslant |a-b| \leqslant |a| + |b|$$

$$\left|\left(\underbrace{2e^{it}}_{a}\right)^3 + \underbrace{1}_{b}\right| \geqslant \left|\left|(2e^{it})^3\right| - |1|\right| \geqslant \left|\left|2^3 e^{3it}\right| - 1\right| = |8-1| =$$
$$= 7 > 0$$

$$\Rightarrow \frac{1}{\left|(2e^{it})^3+1\right|} \leqslant \frac{1}{7}$$

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$$\Rightarrow \left| f(2e^{it}) \right| \leqslant \frac{1}{7} \quad \forall t \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$$

D'où :

$$\left| \int_\gamma f(z)\,dz \right| \leqslant \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{2}{7}\,dt = \frac{2}{7} \times \frac{\pi}{3} = \frac{2\pi}{21}$$

On peut aussi utiliser :

$$\left| \int_\gamma f(z)\,dz \right| \leqslant \ell(\gamma) \cdot \sup \left| f \circ \gamma(t) \right| \quad t \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$$

$$\ell(\gamma) = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2ie^{it}\,dt = \frac{2\pi}{3}$$

$$\left| f \circ \gamma(t) \right| = \left| f(2e^{it}) \right| \leqslant \frac{1}{7} \quad \forall t \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$$

D'où : $\left| \int_\gamma f(z)\,dz \right| \leqslant \frac{2\pi}{21}$

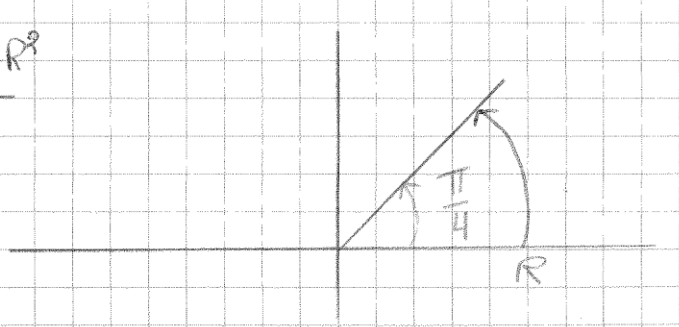
2°/ $\gamma : \left[0, \frac{\pi}{4}\right] \longrightarrow \mathbb{C}$

$t \longrightarrow Re^{it}$ chemin de classe C^1

$$\left| \int_\gamma e^{iz^2}\,dz \right| \leqslant \frac{\pi}{4} \frac{1 - e^{-R^2}}{R}$$

$f : \mathbb{C} \longrightarrow \mathbb{C}$

$z \longrightarrow e^{iz^2}$



f est continue sur $\mathbb{C}$

$$\left| \int_\gamma f(z)\,dz \right| \leqslant \ell(\gamma) \cdot \sup \left| f \circ \gamma(t) \right| \quad t \in \left[0, \frac{\pi}{4}\right]$$

$$\int_\gamma f(z)\,dz = \int_0^{\frac{\pi}{4}} f\circ\gamma(t)\,\gamma'(t)\,dt$$

$$\int_\gamma f(z)\,dz = \int_0^{\frac{\pi}{4}} f(Re^{it})\,Rie^{it}\,dt \leqslant \int_0^{\frac{\pi}{4}} |f(Re^{it})|\,|Rie^{it}|\,dt$$

$$\leqslant R\int_0^{\frac{\pi}{4}} |f(Re^{it})|\,dt$$

$$f(Re^{it}) = \exp\left(i\,(Re^{it})^2\right) = \exp\left(iR^2e^{2it}\right)$$

$$f(Re^{it}) = \exp\left(iR^2(\cos 2t + i\sin 2t)\right)$$

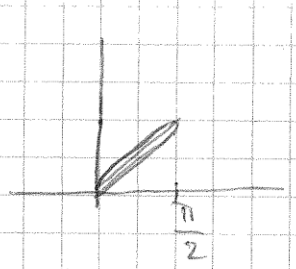
$$= \exp\left(iR^2\cos 2t - R^2\sin 2t\right)$$

$$= \exp\left(iR^2\cos 2t\right)\cdot\exp\left(-R^2\sin 2t\right)$$

$$|f(Re^{it})| = \left|\exp\left(iR^2\cos 2t\right)\cdot\exp\left(-R^2\sin 2t\right)\right|$$

$$= \exp\left(-R^2\sin 2t\right)$$

indication : $\sin x \geqslant \frac{2x}{\pi} \quad \forall\, x \in \left[0, \frac{\pi}{2}\right]$



si $t \in \left[0, \frac{\pi}{4}\right] \Rightarrow x = 2t \in \left[0, \frac{\pi}{2}\right]$

$$\Rightarrow \sin 2t \geqslant \frac{2}{\pi}\,2t = \frac{4t}{\pi}$$

$$\Rightarrow -R^2\sin 2t \leqslant -R^2\frac{4}{\pi}t$$

$$\exp\left(-R^2\sin 2t\right) \leqslant \exp\left(-\frac{4}{\pi}R^2t\right)$$

$$\Rightarrow |f(Re^{it})| \leqslant \exp\left(-\frac{4}{\pi}R^2t\right) \quad \forall\, t \in \left[0, \frac{\pi}{4}\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} |f(Re^{it})|\,dt \leqslant \int_0^{\frac{\pi}{4}} \exp\left(-\frac{4}{\pi}R^2t\right)dt$$

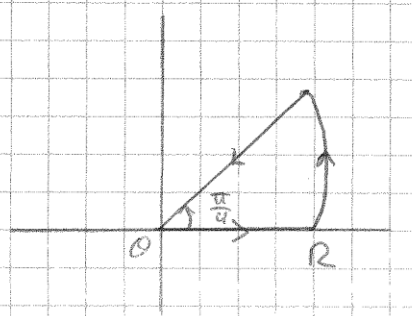
$$\Rightarrow \int_0^{\frac{\pi}{4}} \left| f(Re^{it}) \right| dt \leqslant \left[\frac{\exp\left(-\frac{4}{\pi} R^2 t\right)}{-\frac{4}{\pi} R^2} \right]_0^{\frac{\pi}{4}} = \left[\frac{1 - e^{-R^2}}{\frac{4}{\pi} R^2} \right]$$

D'où : $\left| \int_\gamma f(z)\, dz \right| \leqslant R \cdot \dfrac{1 - e^{-R^2}}{\frac{4}{\pi} R^2} = \dfrac{\pi}{4} \cdot \dfrac{1 - e^{-R^2}}{R}$

$$\left| \int_\gamma f(z)\, dz \right| \leqslant \frac{\pi}{4} \cdot \frac{1 - e^{-R^2}}{R}$$

comme $\lim\limits_{R \to +\infty} \dfrac{1 - e^{-R^2}}{R} = 0$, donc

$$\lim_{R \to +\infty} \int_\gamma f(z)\, dz = 0$$



Série 2 :

Exercice 1 :

$$\gamma : [0,1] \longrightarrow \mathbb{C}$$
$$t \longrightarrow e^{2i\pi t}$$

. Calculer $\int_\gamma \cos(mz)\, dz$, $m \in \mathbb{Z}$

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i) utiliser les primitives

soit $f : \mathbb{C} \longrightarrow \mathbb{C}$
$z \longrightarrow \cos(mz)$

$f(z) = \cos(mz) = \dfrac{e^{imz} + e^{-imz}}{2}$ f est continue sur $\mathbb{C}$ car en fait f est holomorphe sur $\mathbb{C}$.

*) Si $m = 0$, $f(z) = 1$, $\forall z \in \mathbb{C}$

$F : \mathbb{C} \longrightarrow \mathbb{C}$
$z \longrightarrow z$ est holomorphe

et $F'(z) = 1 = f(z) \quad \forall z \in \mathbb{C}$

Donc F est une primitive de f sur $\mathbb{C}$. γ est un chemin de classe C^1, donc

$$\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0))$$

or $\gamma(0) = 1 = e^{2i\pi} = \gamma(1)$

donc $\int_\gamma f(z)\,dz = F(1) - F(1) = 0$

(**) $m \in \mathbb{Z}^*$

$$F : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longrightarrow \frac{\sin(mz)}{m}$$

F est holomorphe sur $\mathbb{C}$ et

$$F'(z) = m \cdot \frac{\cos(mz)}{m}$$

$\forall z \in \mathbb{C} \quad f(z) = \cos(mz)$

Donc F est une primitive de f sur $\mathbb{C}$

$$\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0))$$
$$1 = \gamma(1) = \gamma(0)$$

$$\Rightarrow \int_\gamma f(z)\,dz = F(1) - F(1) = 0$$

Finalement : $\int_\gamma \cos(mz)\,dz = 0 \quad \forall m \in \mathbb{Z}$

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ii) utiliser le théorème de Cauchy :

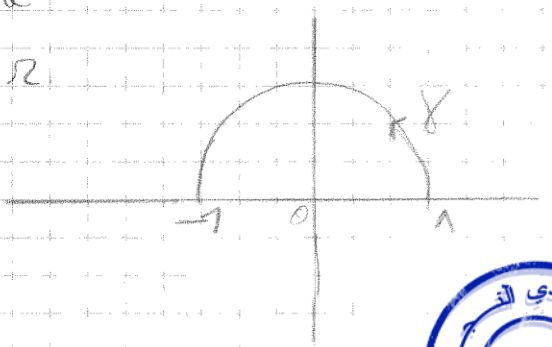
$f : \Omega = \mathbb{C} \longrightarrow \mathbb{C}$

$z \longrightarrow \cos(nz)$ est holomorphe sur $\mathbb{C} = \Omega$

$\gamma : [0,1] \longrightarrow \mathbb{C} = \Omega$ est un chemin fermé, de classe C^1

$\Omega = \mathbb{C}$ est un ouvert étoilé

car $\operatorname{Int} \gamma = D(0,1) \subset \mathbb{C} = \Omega$



Le théorème de Cauchy

$\Rightarrow \int_\gamma f(z)\, dz = 0$

$\Rightarrow \int_\gamma \cos(nz)\, dz = 0 \quad \forall\, n \in \mathbb{Z}$

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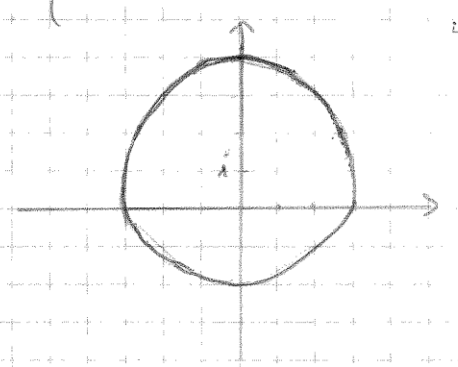
$g : \mathbb{C}^* \longrightarrow \mathbb{C}$

$z \longrightarrow \frac{1}{z}$ continue, mais n'admet pas de primitive sur $\mathbb{C}^*$

Exercice 2 :

a) $\int_{C(i,3)} \frac{\sin z}{z-i}\, dz \qquad C(i,3) = \{ z \in \mathbb{C} \,/\, |z-i| = 3 \}$



$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$

$t \longrightarrow i + 3e^{it}$

soit $f : \mathbb{C} \longrightarrow \mathbb{C}$

$z \longrightarrow \sin z$

f est holomorphe sur $\mathbb{C}$,

soit $\Omega = D(i, 4)$ disque ouvert de centre i et de rayon 4.

$$\gamma : [0, 2\pi] \longrightarrow \Omega$$
$$t \longrightarrow i + 3e^{it}$$

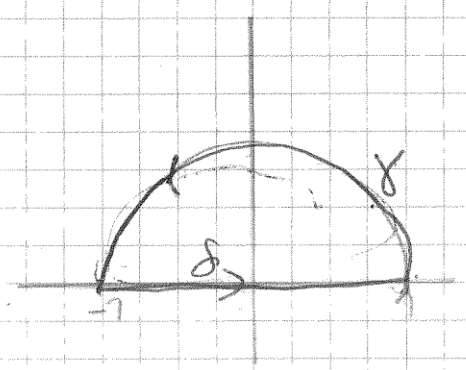
est un chemin de classe C^1 contenue dans Ω, f est holomorphe

La formule de Cauchy $\Rightarrow$ $f(i) = \frac{1}{2\pi i} \int_{C(i,3)} \frac{f(z)}{z-i} dz$

$$\Rightarrow \int_{C(i,3)} \frac{\sin z}{z-i} dz = 2\pi i \sin i$$
$$= 2\pi i \times \frac{e^{-1} - e^{1}}{2i}$$
$$= \pi\left(\frac{1}{e} - e\right).$$

Exercice 1 :

$$\gamma : [0,1] \longrightarrow \mathbb{C}$$
$$t \longrightarrow e^{i\pi t}$$



i) $f : \mathbb{C} \longrightarrow \mathbb{C}$
$$z \longrightarrow \cos(mz)$$

f est continue sur $\mathbb{C}$,

*) $m = 0$; $f(z) = 1$, soit

$$F : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longrightarrow z$$

F est dérivable sur $\mathbb{C}$ et :

$$F'(z) = 1 = f(z)$$

donc F est une primitive de f sur $\mathbb{C}$. Par suite :

$$\int_\gamma f(z)\, dz = F(\gamma(1)) - F(\gamma(0))$$

$\gamma(0) = 1$, $\gamma(1) = -1$

$\int_\gamma f(z)\,dz = F(-1) - F(1) = -2$

**) $m \in \mathbb{Z}^*$, $f(z) = \cos(mz)$

soit $F : \mathbb{C} \longrightarrow \mathbb{C}$

$z \longrightarrow \frac{\sin(mz)}{m}$

F est holomorphe sur $\mathbb{C}$ et $F'(z) = \cos(mz) = f(z)$ $\forall z \in \mathbb{C}$

Donc F est une (fonction) primitive de f sur $\mathbb{C}$

D'où $\int_\gamma f(z)\,dz = F(\gamma(1) - F(\gamma(0))$

$\Rightarrow \int_\gamma \cos(mz)\,dz = \frac{\sin(-m)}{m} - \frac{\sin(m)}{m} = -\frac{2\sin(m)}{m}$

ii)

utiliser le théorème de Cauchy

soit $f : \mathbb{C} \longrightarrow \mathbb{C}$

$z \longrightarrow \cos(mz)$

f est holomorphe sur $\mathbb{C}$,

soit $\Gamma = \gamma \cup \delta : [-1,1] \longrightarrow \mathbb{C}$

$\delta : [-1,0] \longrightarrow \mathbb{C}$

$t \longrightarrow at + b$

$\delta(-1) = -1$ point de départ , $\delta(0) = 1$ point d'arrivée

$\Rightarrow b = 1,\ a = 2$

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$$\delta : [-1,0] \longrightarrow \mathbb{C}$$
$$t \longrightarrow 2t+1$$

$$\Gamma : [-1,1] \longrightarrow \mathbb{C}$$
$$t \longrightarrow \begin{cases} \gamma(t) & \text{si } t \in [0,1] \\ \delta(t) & \text{si } t \in [-1,0] \end{cases}$$

Γ est un chemin fermé, de classe C^1 par morceaux dans $\mathbb{C}$. De plus $\mathbb{C}$ est étoilé, le théorème de Cauchy $\Rightarrow \int_\Gamma f(z)\,dz = 0$

Or: $\int_\Gamma f(z)\,dz = \int_\gamma f(z)\,dz + \int_\delta f(z)\,dz$.

$$\int_\gamma f(z)\,dz = -\int_\delta f(z)\,dz$$

Or: $\int_\delta f(z)\,dz = \int_{-1}^{0} f(\delta(t))\,\delta'(t)\,dt$

$$\int_\delta f(z)\,dz = \int_{-1}^{0} \cos(\overbrace{2mt+m}^{m(2t+1)})\,2\,dt$$

*) $m=0$

$$\int_\delta f(z)\,dz = \int_{-1}^{0} 2\,dt = 2\,[t]_{-1}^{0} = 2$$

$$\Rightarrow \int_\gamma f(z)\,dz = -2$$

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$\cancel{\alpha}$ α) $m \in \mathbb{Z}^*$

$$\int_\gamma f(z)\,dz = \int_{-1}^{0} 2\cos(2mt+m)\,dt$$

$$= \cancel{2}\left[\frac{\sin(2mt+m)}{\cancel{2}m}\right]_{-1}^{0}$$

$$= \frac{\sin(m)}{m} - \frac{\sin(-m)}{m} = \frac{2\sin m}{m}$$

$$\Rightarrow \int_\gamma f(z)\,dz = -\frac{2\sin m}{m}$$

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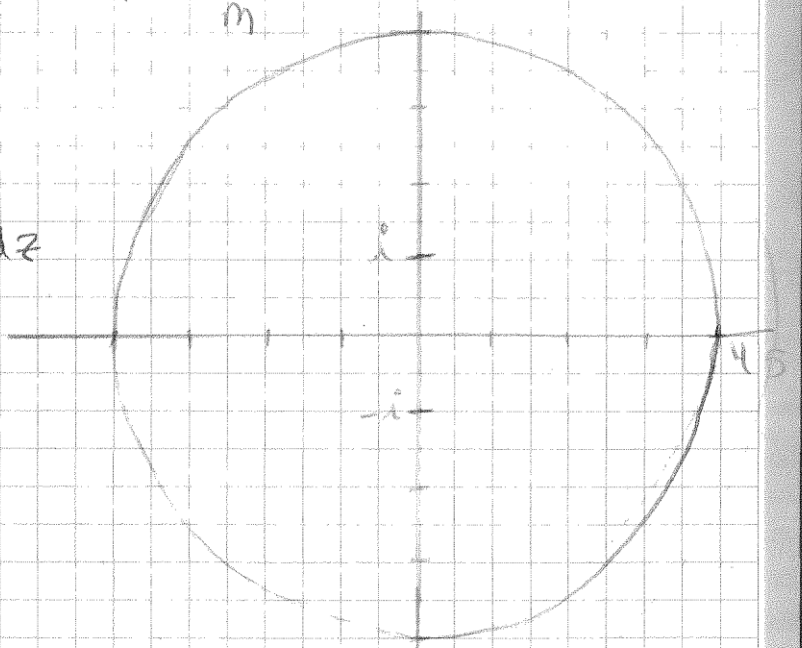
$$\Rightarrow \int_\gamma \cos(mz)\,dz = -2\,\frac{\sin(m)}{m}$$

Exercice 2 :

b) $$\int_{C(0,4)} \frac{\cos z}{z^2 - i^2}\,dz$$



$$z^2 - i^2 = (z-i)(z+i)$$

$$= (z+i)(z-i)$$

Rappel :

f holomorphe sur Ω, $a \in \Omega$

$r > 0$ tq $\overline{D}(a,r) \subset \Omega$

$$f(z) = \frac{1}{2\pi i}\int_{C(a,r)} \frac{f(\xi)}{\xi - z}\,d\xi \qquad |z-a| < r$$

$$\frac{\cos z}{z^2 - i^2} = \frac{\cos z}{(z-i)(z+i)}$$

$$\frac{1}{(z-i)(z+i)} = \frac{a}{z-i} + \frac{b}{z+i}$$

$$\Rightarrow \quad a = \frac{1}{2i} \quad , \quad b = -\frac{1}{2i} \Rightarrow \frac{1}{(z-i)\,(z+i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

$$\Leftrightarrow \quad \frac{\cos z}{(z-i)(z+i)} = \frac{\cos z}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

$$\Leftrightarrow \quad \frac{\cos z}{(z-i)(z+i)} = \frac{1}{2i}\left(\frac{\cos z}{z-i} - \frac{\cos z}{z+i}\right)$$

$$\Leftrightarrow \quad \int_{C(0,4)} \frac{\cos z}{z^2 - i^2}\, dz = \int_{C(0,4)} \frac{1}{2i}\left(\frac{\cos z}{z-i} - \frac{\cos z}{z+i}\right) dz$$

$$= \frac{1}{2i}\int_{C(0,4)} \frac{\cos z}{z-i}\, dz \; \frac{1}{2i}\int_{C(0,4)} \frac{\cos z}{z+i}\, dz$$

soit $\Omega = D(0,5) \subset \Omega$,

$\bar{D}(0,4) = \{ z \in \mathbb{C} \,/\, |z| \leq 4 \} \subset \Omega$

soit $f : \Omega \longrightarrow \mathbb{C}$

$z \longrightarrow \cos z$

f est holomorphe sur Ω,

$i \in D(0,4)$ car $|i-0| = 1 < 4$

$-i \in D(0,4)$ car $|-i-0| = 1 < 4$.

La formule de Cauchy (integrale)

$$\Rightarrow \quad f(i) = \frac{1}{2\pi i} \int_{C(0,4)} \frac{f(\xi)}{\xi - i} \, d\xi \, ,$$

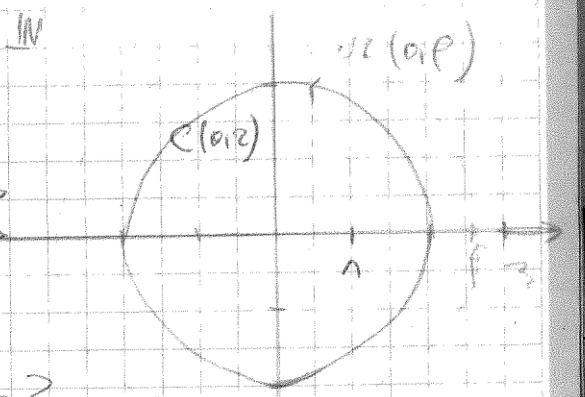
$$f(-i) = \frac{1}{2\pi i} \int_{C(0,4)} \frac{f(\xi)}{\xi + i} \, d\xi$$

Donc: $$\int_{C(0,4)} \frac{\cos z}{z^2 - i^2} \, dz = \pi \, f(i) - \pi \, f(-i)$$

$$= \pi \left(\cos i - \cos(-i) \right)$$

$$= \pi \left(\cos i - \cos i \right) = 0 \qquad 2 < \rho < 3$$

c) $$\int_{C(0,2)} \frac{dz}{(z-1)^m (z-3)} \, , \quad m \in \mathbb{N}$$



$$\Omega = D(0,\rho) = \{ z \in \mathbb{C} \,/\, |z-0| < \rho \}$$
$$(\rho = 2.5)$$

On a: $\bar{D}(0,2) = \{ z \in \mathbb{C} \,/\, |z| \leq 2 \} \subset \Omega$

soit $f : \Omega \longrightarrow \mathbb{C}$

$$z \longrightarrow \frac{1}{z-3}$$

f est holomorphe sur Ω,

la formule de cauchy $\Rightarrow$

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$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{C(0,2)} \frac{f(\xi)}{(\xi - z)^{m+1}}\, d\xi \qquad |z| < 2$$

$$\forall\, m \in \mathbb{N}$$

*) Si $m = 0$

$$\int_{C(0,2)} \frac{dz}{z-3} = \int_{C(0,2)} f(z)\, dz = 0$$ d'après le théorème de Cauchy.

**) Si $m \in \mathbb{N}^*$, posons $m+1 = m$

c.à.d, $m = m-1$

On obtient :

$$f^{(m-1)}(1) = \frac{(m-1)!}{2\pi i}\int_{C(0,2)} \frac{f(\xi)}{(\xi - z)^{m-1+1}}\, d\xi \quad \text{car } |1| < 2.$$

$$\Rightarrow \quad f^{(m-1)}(1) = \frac{(m-1)!}{2\pi i}\int_{C(0,2)} \frac{d\xi}{(\xi-1)^m(\xi-3)}$$

donc :

$$\int_{C(0,2)} \frac{dz}{(z-1)^m(z-3)} = \frac{2\pi i}{(m-1)!}\, f^{(m-1)}(1)$$

$$f(z) = \frac{1}{z-3}$$

$$f'(z) = -\frac{1}{(z-3)^2} = \frac{(-1)^1\, 1!}{(z-3)^{1+1}}$$

$$f''(z) = \frac{2}{(z-3)^3} = \frac{(-1)^2\, 2!}{(z-3)^{2+1}}$$

$$f^{(m)}(z) = \frac{(-1)^m\, m!}{(z-3)^{m+1}} \qquad \forall z \in \mathbb{C} \setminus \{3\}$$

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par récurence sur $m \in \mathbb{N}$

$m = 0 \qquad f(z) = \dfrac{1}{z-3}$

$m = 1 \qquad f'(z) = -\dfrac{1}{(z-3)^2}$

supposons : $\quad f^{(m)}(z) = \dfrac{(-1)^m \, m!}{(z-3)^{m+1}}$

$$f^{(m+1)}(z) = \left(f^{(m)}\right)'(z) = (-1)^m \, m! \; \frac{-(m+1)(z-3)^m}{(z-3)^{2m+2}}$$

$$f^{(m+1)}(z) = \frac{(-1)^{m+1}(m+1)!}{(z-3)^{m+1+1}}$$

Cq : $\quad f^{(m)}(z) = \dfrac{(-1)^m \, m!}{(z-3)^{m+1}} \qquad \forall z \in \mathbb{C} \setminus \{3\}$

donc : $\quad f^{(m-1)}(1) = \dfrac{(-1)^{m-1}(m-1)!}{(1-3)^m} = \dfrac{(-1)^{m-1}(m-1)!}{(-1)^m 2^m}$

$$= -\frac{(m-1)!}{2^m}$$

D'où :

$$\int_{C(0,2)} \frac{dz}{(z-1)^m (z-3)} = \frac{-2\pi i}{(m-1)!} \; \frac{(m-1)!}{2^m}$$

$$= \frac{-2\pi i}{2^m} = -\frac{\pi i}{2^{m-1}}$$

$$\int_{C(0,2)} \frac{dz}{(z-1)^m (z-3)} = \frac{-\pi i}{2^{m-1}} \qquad \forall m \in \mathbb{N}^*$$

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Exercice 3 :

$$I = \int_0^{+\infty} \cos(x^2)\,dx \quad , \quad J \int_0^{+\infty} \sin(x^2)\,dx$$

$$I + iJ = \int_0^{+\infty} \cos(x^2)\,dx + i\int_0^{+\infty} \sin(x^2)\,dx.$$

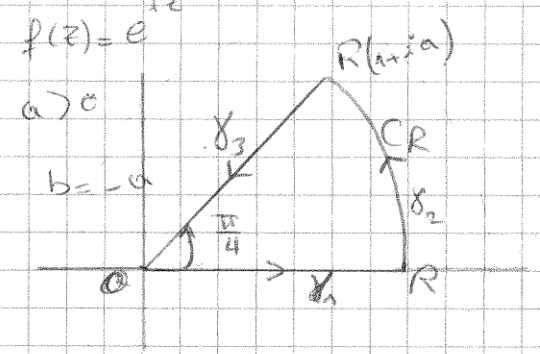
$$= \int_0^{+\infty} \left(\cos(x^2) + i\left(\sin(x^2)\right)\right) dx$$

$$= \int_0^{+\infty} e^{ix^2}\,dx.$$

$$I + iJ = \int_0^{+\infty} e^{ix^2}\,dx.$$

soit : $f(z) = e^{iz^2}$ et le contour $\rho > R\sqrt{1+a^2}$

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$$



soit $\Omega = D(0, \rho)$, $\rho > R$

$$f : \Omega \longrightarrow \mathbb{C}$$
$$z \longrightarrow e^{iz^2}$$

f est holomorphe sur Ω,

γ est de classe C^1 par morçeaux fermé conten dans Ω (Ω est un disque) Le théorème de Cauchy $\Rightarrow \int_\gamma f(z)\,dz = 0$

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz.$$

*) $$\int_{\gamma_1} f(z)\,dz = \int_0^R f\circ\gamma_1(t)\,\gamma_1'(t)\,dt = \int_0^R f(t)\,dt.$$

$$\gamma_1 : [0,R] \longrightarrow \mathbb{C}$$
$$t \longrightarrow t$$

$$\int_{\gamma_1} f(z)\,dz = \int_0^R e^{it^2}\,dt$$

*) $$\gamma_3 : [0,R] \longrightarrow \mathbb{C}$$
$$t \longrightarrow (R-t)\,e^{i\frac{\pi}{4}}$$

$$\gamma(0) = R e^{i\frac{\pi}{4}}, \quad \gamma_3(R) = 0$$

$$\int_{\gamma_3} f(z)\,dz = \int_0^R f\circ\gamma_3(t)\,\gamma_3'(t)\,dt \qquad \left(e^{i\frac{\pi}{4}}\right)^2 = e^{i\frac{\pi}{2}} = i$$

$$= \int_0^R f\left((R-t)e^{i\frac{\pi}{4}}\right)\left(-e^{i\frac{\pi}{4}}\right)dt$$

$$= -e^{i\frac{\pi}{4}}\int_0^R \exp\left(i(R-t)^2\left(e^{i\frac{\pi}{4}}\right)^2\right)dt$$

$$\int_{\gamma_3} f(z)\,dz = -e^{i\frac{\pi}{4}}\int_0^R \exp\left(-(R-t)^2\right)dt$$

$$s = R-t \Rightarrow ds = -dt.$$

$$\int_{\gamma_3} f(z)\,dz = -e^{i\frac{\pi}{4}}\int_R^0 \exp(-s^2)\,(-ds)$$

$$= -e^{i\frac{\pi}{4}}\int \exp(-s^2)\,ds$$

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$$\int_{\gamma_3} f(z)\,dz = -e^{i\frac{\pi}{4}} \int_0^R e^{-s^2}\,ds.$$

$$(\star\star\star) \quad \left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{\pi}{4}\,\frac{1-e^{-R^2}}{R} \xrightarrow[R\to+\infty]{} 0$$

$$\Rightarrow \lim_{R\to+\infty} \int_{\gamma_2} f(z)\,dz = 0$$

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$(\star)$ devient :

$$(\star\star) \quad \int_0^R e^{it^2}\,dt + \int_{\gamma_2} f(z)\,dz - e^{i\frac{\pi}{4}} \int_0^R e^{-s^2}\,ds = 0 \quad \forall R > 0.$$

Faisons tendre $R \to +\infty$ dans $(\star\star)$

$$\lim_{R\to+\infty} \int_0^R e^{it^2}\,dt = \int_0^{+\infty} e^{it^2}\,dt = I + iJ$$

$$\lim_{R\to+\infty} \int_0^R e^{-s^2}\,ds = \int_0^{+\infty} e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2}$$

$$I + iJ + 0 - e^{i\frac{\pi}{4}} \times \frac{\sqrt{\pi}}{2} = 0$$

$$I + iJ = e^{i\frac{\pi}{4}} \times \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\begin{cases} I = \dfrac{\sqrt{\pi}}{2}\cos\dfrac{\pi}{4} = \dfrac{\sqrt{\pi}}{2} \times \dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2\pi}}{4} \\ J = \dfrac{\sqrt{\pi}}{2}\sin\dfrac{\pi}{4} = \dfrac{\sqrt{\pi}}{2}\,\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2\pi}}{4} \end{cases}$$

Exercice 5 : Points singuliers et leurs natures

a) $f(z) = \dfrac{z^4}{(z^4+16)^2}$

$$D_f = \{ z \in \mathbb{C} / z^4 + 16 \neq 0 \}$$

$$z^4 + 16 = 0 \Leftrightarrow z^4 = -16$$

$$\Leftrightarrow z^4 = 16 e^{i\pi}$$

$$\Leftrightarrow z_k = \sqrt[4]{16}\, e^{i\frac{\pi}{4} + i\frac{2k\pi}{4}} \quad , \quad k = 0, 1, 2, 3,$$

$$z_k = 2 e^{i\frac{\pi}{4} + ik\frac{\pi}{2}}$$

$$z_0 = 2 e^{i\frac{\pi}{4} + i\frac{\pi}{2} \times 0} = 2 e^{i\frac{\pi}{4}}$$

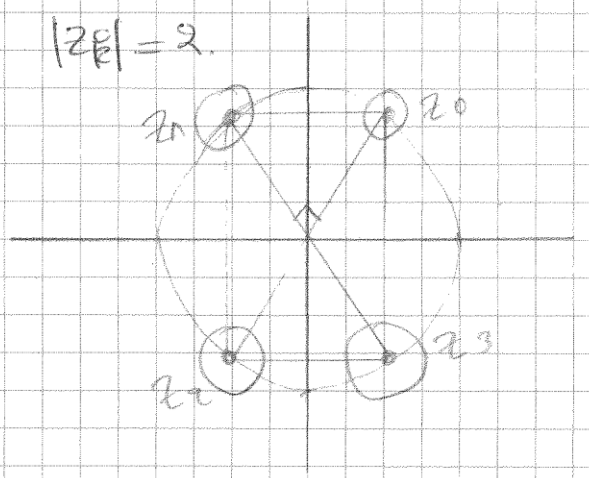
$$z_1 = 2 e^{i\frac{\pi}{4} + i\frac{\pi}{2}} = 2 e^{\frac{3i\pi}{4}}$$

$$z_2 = 2 e^{i\frac{\pi}{4} + i\pi} = 2 e^{\frac{i5\pi}{4}}$$

$$z_3 = 2 e^{i\frac{\pi}{4} + \frac{3i\pi}{2}} = 2 e^{\frac{7i\pi}{4}}$$

z_0, z_1, z_2, z_3 sont des distinctions deux à deux,

$|z_k| = 2$.



$a = \sqrt[n]{b}$ $\quad \xi_k = e^{\frac{i2k\pi}{n}}$

$z^n + a = 0$ $\quad \xi^n = 1$ $\quad 0 \leq k \leq n-1$

$z^n - a = 0$ $\quad \xi = \dfrac{z}{\sqrt[n]{a}}$, $\xi_k = e^{i\frac{\pi}{n} + \frac{i2k\pi}{n}}$

$\Downarrow$ $\quad \xi^n = 1$

$$\left(\frac{z}{\sqrt[n]{a}}\right)^n = 1$$

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$D_f = \mathbb{C} \setminus \{z_0, z_1, z_2, z_3\}$

les points singuliers isolées de f sont z_0, z_1, z_2, z_3.

$z^4 + 16 = (z - z_0)(z - z_1)(z - z_2)(z - z_3)$.

$$f(z) = \frac{z^4}{(z-z_0)^2(z-z_1)^2(z-z_2)^2(z-z_3)^2}$$

Natures des points singuliers :

*) z_0 : $\lim_{z \to z_0} (z - z_0)^2 f(z) = \lim_{z \to z_0} \frac{z^4}{(z-z_1)^2(z-z_2)^2(z-z_3)^2}$

$$= \frac{z_0^4}{(z_0-z_1)^2(z_0-z_2)^2(z_0-z_3)^2}$$

$$= \frac{-16}{(\ )(\ )(\ )} \neq 0$$

donc z_0 est un pôle d'ordre $m = 2$ de f

**) m̂ chose pour z_1, z_2, z_3 se sont des pôles d'ordre $m = 2$.

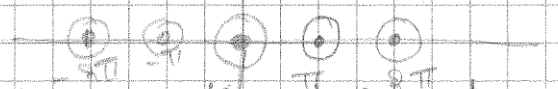
b) $f(z) = \frac{z^8 - \pi^2}{\sin^2 z}$

Points singuliers de f :

$D_f = \{ z \in \mathbb{C} / \sin z \neq 0 \}$

$\sin z = 0 \iff z_k = k\pi, \ k \in \mathbb{Z}$

$D_f = \mathbb{C} - \{ k\pi \,/\, k \in \mathbb{Z} \}$



Les points singuliers isolés de f sont $z_k = k\pi$, $k \in \mathbb{Z}$

Nature des points singuliers :

$$\lim_{z \to z_k} (z - z_k)\, f(z)$$

Posons $w = z - z_k \longrightarrow 0$.

$z \longrightarrow z_k$

$$g(w) = f(z) = f(w + z_k) = \frac{(w + z_k)^2 - \pi^2}{\sin^2(w + z_k)}$$

$$\sin(w + z_k) = \sin(w + k\pi) = \sin w \cos k\pi$$

$$= (-1)^k \sin w$$

$$\sin^2(w + z_k) = \left((-1)^k \sin w\right)^2 = \sin^2 w$$

$$\cancel{g(w) = \frac{(k\pi + w)^2 - \pi^2}{}}$$

$$g(w) = \frac{(w + z_k)^2 - \pi^2}{\sin^2 w}$$

$$\lim_{w \to 0} w^2 g(w) = \lim_{w \to 0} w^2 \frac{(w + z_k)^2 - \pi}{\sin^2 w}$$

or $\lim_{w \to 0} \frac{\sin w}{w} = 1 \Rightarrow \lim_{w \to 0} \frac{w^2}{\sin^2 w} = 1$.

$$\Rightarrow \lim_{w \to 0} w^2 g(w) = (k\pi)^2 - \pi^2 = (k^2 - 1)\pi^2.$$

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$$\Rightarrow \lim_{z \to z_k} (z - z_k)^2 f(z) = \lim_{w \to 0} w^2 g(w) = (k^2 - 1)\pi^2$$

i) si $k^2 - 1 \neq 0 \Rightarrow k \neq \mp 1$.

on a : $\lim_{z \to z_k} (z - z_k)^2 f(z) = (k^2 - 1)\pi^2 \neq 0$

donc $z_k = k\pi$ est un pôle d'ordre 2.

ii) $k^2 = 1 \Leftrightarrow k = 1$ ou $k = -1$.

* Nature $z_1 = \pi$.

$$\lim_{z \to z_1} (z - z_1) f(z) = \lim_{w \to 0} w \frac{(w+\pi)^2 - \pi^2}{\sin^2 w}$$

$$(w+\pi)^2 - \pi^2 = (w + \pi - \pi)(w + \pi + \pi) = w(w + 2\pi)$$

$$\Rightarrow \lim_{z \to z_1} (z - z_1) f(z) = \lim_{w \to 0} \frac{w^2 (w + 2\pi)}{\sin^2 w}$$

$$= 2\pi \neq 0$$

$\Rightarrow z_1$ est un pôle de multiplicité 1.

* Nature $z_{-1} = -\pi$.

$$\lim_{z \to z_{-1}} (z - z_{-1}) f(z) = \lim_{w \to 0} w \frac{(w - \pi)^2 - \pi^2}{\sin^2 w}$$

$$(w - \pi)^2 - \pi^2 = (w - \pi - \pi)(w - \pi + \pi)$$

$$= w(w - 2\pi)$$

$$\Rightarrow \lim_{z \to z_{-1}} (z - z_{-1}) f(z) = \lim_{w \to 0} \frac{w^2 (w - 2\pi)^2 \cancel{\pi^2}}{\sin^2 w} = -2\pi \neq 0$$

$\Rightarrow z_{-1} = -\pi$ est un pôle simple.

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c) : $f(z) = \dfrac{1 - \cos z}{\sin z}$

$D_f = \{ z \in \mathbb{C} / \sin z \neq 0 \}$

$= \mathbb{C} - \{ k\pi / k \in \mathbb{Z} \}$.

Les points singuliers isolées de f sont $z_k = k\pi$, $k \in \mathbb{Z}$.

Natures des points :

$$\lim_{z \to z_k} (z - z_k) f(z) =$$

Posons $w = z - z_k \xrightarrow[z \to z_k]{} 0$

$g(w) = f(z) = f(w + z_k)$

$$\cos(w + z_k) = \cos(w + k\pi)$$
$$= \cos w \cos(k\pi) - \sin w \sin(k\pi)$$
$$= (-1)^k \cos w$$

$$\sin(w + z_k) = \sin(w + k\pi) = \sin w \cos k\pi$$
$$= (-1)^k \sin w.$$

$$g(w) = \frac{1 - (-1)^k \cos w}{(-1)^k \sin w}$$

$\neq 0$

$$\lim_{w \to 0} w\, g(w) = \lim_{w \to 0} \cancel{w} \frac{1 - (-1)^k \cos w}{(-1)^k \frac{\sin w}{w} \times \cancel{w}} = \frac{1 - (-1)^k}{(-1)^k}$$

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i) k impair, $k = 2p+1$

$$\lim_{w \to 0} w\, g(w) = \frac{1+1}{-1} = -2 \neq 0$$

$$\Rightarrow \lim_{z \to z_k} (z - z_k)\, f(z) = -2 \neq 0$$

$\Rightarrow z_{2p+1} = (2p+1)\pi$ est un pôle d'ordre 1.

ii) k pair, $k = 2p$

$$\lim_{w \to 0} g(w) = \lim_{w \to 0} w^2 \frac{1 - \cos w}{w^2 \sin w}$$

$$= \lim_{w \to 0} w \frac{1 - \cos w}{w^2} \times \frac{w}{\sin w} = 0$$

$\Rightarrow \lim_{z \to z_{2p}} f(z) = 0$ existe finie.

$\Rightarrow z_{2p} = 2p\pi$ est un point singulier éliminable.

d) $f(z) = \dfrac{1}{\cos \frac{1}{z}}$

$$D_f = \left\{ z \in \mathbb{C} \,/\, z \neq 0 \text{ et } \cos \frac{1}{z} \neq 0 \right\}$$

$$\cos w = 0 \iff w_k = (2k+1)\frac{\pi}{2}, \; k \in \mathbb{Z}$$

$$\left.\begin{array}{l} \cos \frac{1}{z} = 0 \\ z \neq 0 \end{array}\right\} \iff \frac{1}{z} = (2k+1)\frac{\pi}{2} \text{ et } z \neq 0$$

$$\iff z_k = \frac{2}{(2k+1)\pi}, \; k \in \mathbb{Z}$$

$$D_f = \left\{ z \in \mathbb{C} \,/\, z = 0 \text{ ou } z = \frac{2}{(2k+1)\pi}, \; k \in \mathbb{Z} \right\}$$

Les points singuliers isolés de f sont $z=0$ et $z_k = \frac{2}{(2k+1)\pi}$, $k \in \mathbb{Z}$

Nature des points singuliers :

*) $z_k = \frac{2}{(2k+1)\pi}$, $k \in \mathbb{Z}$.

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$$\lim_{z \to z_k} (z - z_k) f(z) = \lim_{z \to z_k} (z - z_k) \times \frac{1}{\cos \frac{1}{z}}$$

Posons $g(z) = \cos \frac{1}{z}$, $z_k \neq 0$ $\forall k \in \mathbb{Z}$

$$g(z_k) = \cos \frac{1}{z_k} = \cos\left(\frac{\pi(2k+1)}{2}\right) = 0$$

$$\lim_{z \to z_k} \frac{g(z) - g(z_k)}{z - z_k} = g'(z_k).$$

$$g'(z) = -\frac{1}{z^2}\left(-\sin \frac{1}{z}\right)$$

$$g'(z_k) = \frac{1}{z_k^2} \sin\left(\frac{1}{z_k}\right)$$

$$g'(z_k) = \left(\frac{(2k+1)\pi}{2}\right)^2 \sin (2k+1)\frac{\pi}{2}$$

$$= -(2k+1)^2 \frac{\pi^2}{4} \neq 0 \quad \forall k \in \mathbb{Z}$$

$$\lim_{z \to z_k} (z - z_k) f(z) = \lim_{z \to z_k} (z - z_k) \times \frac{1}{g(z_k)}$$

$$\cancel{\lim_{z \to z_k} \frac{z - z_k}{g(z) - g(z_k)} = \lim_{z \to z_k} \frac{z - z_k}{g(z) - g(z_k)}}$$

$$= \lim_{z \to z_k} \frac{1}{\frac{g(z) - g(z_k)}{z - z_k}} = \lim_{z \to z_k} \frac{1}{g'(z_k)}$$

$$\lim_{z \to z_k} (z - z_k) f(z) = \frac{-4}{\pi^2 (2k+1)^2} \neq 0 \quad \forall k \in \mathbb{Z}$$

Donc z_k est un pôle d'ordre 1.

**) $z = 0$

$$\lim_{z \to 0} f(z) \ ?$$

Soit $w_k = \frac{1}{k\pi}$, $k \geq 1$, $\lim_{k \to \infty} w_k = 0$

$$f(w_k) = f\left(\frac{1}{k\pi}\right) = \frac{1}{\cos(k\pi)} = \frac{1}{(-1)^k}$$

$$= (-1)^k$$

$$\lim_{k \to \infty} f(w_k) = \lim_{k \to \infty} (-1)^k \quad \text{n'existe pas.}$$

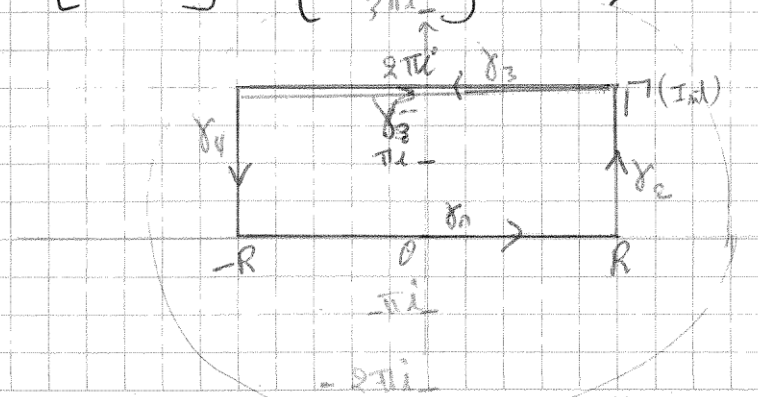
$\Rightarrow \lim_{z \to 0} f(z)$ n'existe pas par consequent $z = 0$

est un pôle (pl) singulier essentiel.

Exercice 6 :

$$a \in]0,1[\quad , I = \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x + 1} dx$$

$$\text{Int}\Gamma = [-R, R] \times [0, 2\pi] \qquad R > 0$$



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$$f(z) = \frac{e^{ax}}{e^{x}+1}$$

$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ est un chemin fermé de classe C^1 par morçeaux.

$$D_f = \{ z \in \mathbb{C} \ / \ e^{z}+1 \neq 0 \}$$

$$e^{z}+1 = 0 \iff e^{z} = -1 = e^{i\pi} = e^{i\pi + 2k\pi i}$$

$$\iff z_k = i\pi + 2k\pi i \ , \ k \in \mathbb{Z}.$$

$$D_f = \mathbb{C} \setminus \{ i\pi + 2k\pi i \ / \ k \in \mathbb{Z} \}.$$

$k \in \mathbb{Z}$

les points singuliers isolés de f sont $z_k = i\pi + 2k\pi i$

On applique le théorème des résidus

soit $\Omega = D(0, \rho)$, $\rho > \sqrt{R^2 + (2\pi)^2}$

$\Gamma \subset \Omega \setminus \{ i\pi + 2k\pi i \ / \ k \in \mathbb{Z} \}$

càd Γ ne passe pas par les points singuliers de f,

Γ est fermé, simple, C^1 par morçeaux.

f est holomorphe sur $\Omega \setminus \{ i\pi + 2k\pi i \ / \ -m \leq k \leq m \}$

avec $|i\pi + 2k\pi i| < \rho$

$$\gamma(t) = z_0 + R e^{i\theta}$$

centre. rayon

$$\iff |(2k+1)|\pi < \rho$$

$$\frac{-\rho}{\pi} < 2k+1 < \frac{\rho}{\pi}$$

$$\iff \frac{1}{2}\left(\frac{-\rho}{\pi} - 1\right) < k < \left(\frac{\rho}{\pi} - 1\right)\frac{1}{2}$$

on choisit $m \in \mathbb{N}$ tq $m = E\left[\left(\frac{\rho}{\pi} - 1\right)\frac{1}{2}\right]$

Le théorème des résidus $\Rightarrow$

$$\int_{\Gamma} f(z)\,dz = 2\pi i \ \mathrm{Res}(f, \pi i)$$

car πi le seule point singulier qui à l'interieur de Γ.

calcul de $\mathrm{Res}(f, \pi i)$

Nature du point $z_0 = \pi i$

$$\lim_{z \to \pi i} (z - \pi i)\, f(z)$$

Posons $P(z) = e^{az}$, $\varphi(z) = e^{z} + 1$

$$P(z_0) = e^{a\pi i} \neq 0$$

$$\varphi(z_0) = e^{\pi i} + 1 = 0$$

$$\varphi'(z) = e^{z}, \quad \varphi'(z_0) = e^{i\pi} = -1 \neq 0$$

$$\lim_{z \to \pi i} (z - i\pi)\, f(z) = \lim_{z \to z_0} (z - \pi i)\, \frac{P(z)}{\varphi(z)}$$

$$= \lim_{z \to z_0} \frac{P(z)}{\dfrac{\varphi(z) - \varphi(\pi i)}{z - \pi i}} = \lim_{z \to \pi i} \frac{P(z)}{\varphi'(z_0)} = \frac{e^{a\pi i}}{-1}$$

$$\lim_{z \to \pi i} (z - z_0)\, f(z) = -e^{a\pi i} \neq 0$$

$\Rightarrow$ $z_0 = \pi i$ est un pôle simple de f.

Donc $\operatorname{Res}(f, \pi i) = \frac{1}{(1-1)!} \lim_{z \to z_0} \left[\frac{d^{1-1}}{dz^{1-1}} (z - \pi i)^1 f(z) \right]$

$$= \lim_{z \to \pi i} (z - \pi i) f(z)$$

$$= -e^{a\pi i}$$

Finalement :

$$\int_\Gamma f(z)\,dz = 2\pi i \left(-e^{a\pi i}\right)$$

$$= -2\pi i e^{a\pi i}$$

Or :

$$\int f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$$

*) $\gamma_1 : [-R, R] \longrightarrow \mathbb{C}$

$t \longrightarrow t$

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} f(t)\,dt = \int_{-R}^{R} \frac{e^{at}}{e^t + 1}\,dt$$

$$\lim_{R \to +\infty} \int_{\gamma_1} f(z)\,dz = \lim_{R \to +\infty} \int_{-R}^{R} \frac{e^{at}}{e^t + 1}\,dt = \int_{-\infty}^{+\infty} \frac{e^{at}}{e^t + 1}\,dt.$$

*) $\gamma_2 : [0, 2\pi] \longrightarrow \mathbb{C}$

$t \longrightarrow R + it$

$$\int_{\gamma_2} f(z)\,dz = \int_0^{2\pi} f(R+it)\,i\,dt = \int_0^{2\pi} f(R+it)\,i\,dt$$

$$= i \int_0^{2\pi} \frac{e^{a(R+it)}}{e^{R+it} + 1}\,dt$$

$$\int_{\gamma_2} f(z)\,dz = i\int_0^{2\pi} \frac{e^{aR}\times e^{iat}}{e^{R}\times e^{it}+1}\,dt$$

$$\left|\int_{\gamma_2} f(z)\,dz\right| = \left|i\int_0^{2\pi} \frac{e^{aR}e^{ita}}{e^{R}\times e^{it}+1}\,dt\right|$$

$$\leqslant \int_0^{2\pi} \left|\frac{e^{aR}\times e^{ita}}{e^{R}\times e^{it}+1}\right| dt = e^{aR}\int_0^{2\pi} \frac{dt}{|e^{R}e^{it}+1|}$$

$$|e^{R}e^{it}+1| \geqslant \big||e^{R}\times e^{it}|-|1|\big| = |e^{R}-1|$$

$$= e^{R}-1 \quad , \quad R>0$$

$$|e^{R}e^{it}+1| \geqslant e^{R}-1 > 0$$

$$\Rightarrow \frac{1}{|e^{R}e^{it}+1|} \leqslant \frac{1}{e^{R}-1} \qquad \forall t\in[0,2\pi].$$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leqslant e^{aR}\int_0^{2\pi} \frac{dt}{e^{R}-1} = \frac{2\pi e^{aR}}{e^{R}-1}$$

$$\Rightarrow \lim_{z\to+\infty} \frac{e^{aR}}{e^{R}-1} = \lim_{z\to+\infty} \frac{e^{(a-1)R}}{1-e^{-R}} = 0 \quad \text{car } a<0$$

$$\Rightarrow \lim_{R\to+\infty} \int_{\gamma_2} f(z)\,dz = 0$$

*) $\gamma_3^- : [-R,R] \longrightarrow \mathbb{C}$

$$t \longrightarrow t+2\pi i$$

$$\int_{\gamma_3} f(z)\,dz = -\int_{\gamma_3^-} f(z)\,dz = -\int_{-R}^{R} f(t+2\pi i)\,dt$$

$$\int_{\gamma_3} f(z)\,dz = -\int_{-R}^{R} \frac{e^{a(t+2\pi i)}}{e^{t+2\pi i}+1}\,dt = -\int_{-R}^{R} \frac{e^{2\pi a i}\times e^{at}}{e^{t}\times e^{2\pi i}+1}\,dt$$

$$= -e^{2\pi a i}\int_{-R}^{R} \frac{e^{at}}{e^{t}+1}\,dt$$

$$\lim_{R\to+\infty}\int_{\gamma_3} f(z)\,dz = -e^{2\pi a i}\int_{-\infty}^{+\infty} \frac{e^{at}}{e^{t}+1}\,dt$$

α) $\gamma_4^- : [0, 2\pi] \longrightarrow \mathbb{C}$

$$t \longrightarrow -R+it$$

$$\int_{\gamma_4} f(z)\,dz = -\int_{\gamma_4^-} f(z)\,dz = -\int_0^{2\pi} f(-R+it)\,i\,dt$$

$$\left|\int_{\gamma_4} f(z)\,dz\right| = \left|-i\int_0^{2\pi} \frac{e^{a(-R+it)}}{e^{-R+it}+1}\,dt\right|$$

$$\leq \int_0^{2\pi} \frac{\left|e^{a(-R+it)}\right|}{\left|e^{-R+it}+1\right|}\,dt$$

$a<0$

$$\leq \int_0^{2\pi} \frac{e^{-aR}\,dt}{\left|e^{-R}-1\right|} = 2\pi\times\frac{e^{-aR}}{1-e^{-R}}$$

$$\lim_{R\to+\infty}\frac{e^{-aR}}{1-e^{-R}} = 0, \text{ donc}$$

$$\lim_{R\to+\infty}\int_{\gamma_4} f(z)\,dz = 0$$

on a : $\displaystyle\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$

$$= -2\pi i e^{a\pi i} \qquad \forall R>0$$

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Faisons tendre $R \longrightarrow$

On obtient :

$$I + 0 - e^{2\pi a i} I + 0 = -2\pi i e^{a\pi i}$$

$$\Rightarrow I\left(1 - e^{2\pi a i}\right) = -2\pi i e^{a\pi i}$$

$$\Rightarrow I = \frac{-2\pi i e^{a\pi i}}{1 - e^{2\pi a i}}$$

$$\Rightarrow I = \frac{2\pi i e^{a\pi i}}{e^{2\pi a i} - 1} = \frac{2\pi i}{e^{\pi a i} - e^{-\pi a i}}$$

$$I = \frac{\pi}{\sin(a\pi)} \qquad a \in]0,1[.$$

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Exercice 10 :

ii) $I_m = \int_0^{2\pi} (\cos t)^{2m}\, dt, \; m \in \mathbb{N}.$

$$I_0 = \int_0^{2\pi} dt = 2\pi$$

soit $m \in \mathbb{N}^*$ Posons $f = e^{it}$

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{z + \frac{1}{z}}{2}$$

$$dz = i e^{it} dt = iz\, dt \Rightarrow dt = \frac{dz}{iz}$$

$$I_m = \int_{C(0,1)} \frac{\left(z + \frac{1}{z}\right)^{2m}}{2} \frac{dz}{iz}$$

$$= \frac{1}{2^{2m}} \frac{1}{i} \int_{C(0,1)} \left(z + \frac{1}{z}\right)^{2m} \frac{dz}{z}$$

$$I_m = \frac{i}{2^{2m}} \int_{C(0,1)} \left(z + \frac{1}{z}\right)^{2m} \frac{dz}{z}$$

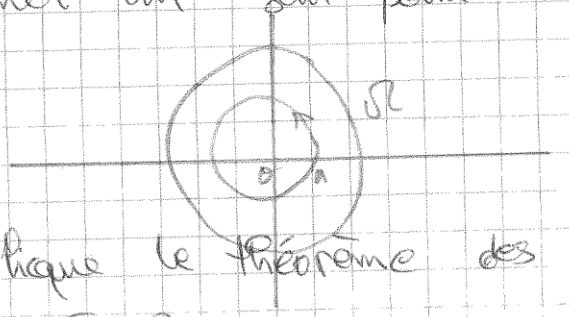
Calculons $\displaystyle\int_{C(0,1)} \left(z+\frac{1}{z}\right)^{2m} \frac{dz}{z}$

Posons $f(z) = \left(z+\frac{1}{z}\right)^{2m} \frac{1}{z}$

$$f(z) = \left(\frac{z^2+1}{z}\right)^{2m} \frac{1}{z} = \frac{(z^2+1)^{2m}}{z^{2m+1}}$$

$$D_f = \{ z \in \mathbb{C} / \ z^{2m+1} \neq 0 \} = \mathbb{C}^*$$

f admet un seul point singulier $z_0 = 0$



$\rho > 1$

on applique le théorème des résidus soit $\Omega = D(0,\rho)$

$$f : \Omega - \{0\} \longrightarrow \mathbb{C}$$
$$z \longrightarrow \frac{(z^2+1)^{2m}}{z^{2m+1}}$$

f est holomorphe sur $\Omega - \{0\}$.

$C(0,1) \subset \Omega \setminus \{0\}$, $C(0,1)$ est un contour fermée, simple, de classe C^1. Le théorème des résidus

$$\Rightarrow \int_{C(0,1)} f(z)\,dz = 2\pi i \, \mathrm{Res}(f,0)$$

- Calcul de $\mathrm{Res}(f,0)$

Nature de $z_0 = 0$

$$\lim_{z\to 0} (z-0)^{2m+1} f(z) = \lim_{z\to 0} (z^2+1)^{2m} = 1 \neq 0$$

donc $z_0 = 0$ est un pôle d'ordre $m = 2m+1$

Posons $g(z) = z^{2m+1} f(z) = (z^2+1)^{2m}$

$$\mathrm{Res}(f,0) = \frac{1}{(2m)!} \lim_{z\to 0} \frac{d^{2m}}{dz^{2m}} g(z)$$

$$\mathrm{Res}(f,0) = \frac{1}{(2m)!} g^{(2m)}(0)$$

Autre manière de calcul :

$$f(z) = \frac{1}{z}\left(z+\frac{1}{z}\right)^{2m}, \quad z \in \mathbb{C}^*$$

$$(a+b)^{2m} = \sum_{k=0}^{2m} C_{2m}^{k} a^{k} b^{2m-k}$$

$$f(z) = \frac{1}{z} \sum_{k=0}^{2m} C_{2m}^{k} z^{k} \left(\frac{1}{z}\right)^{2m-k}$$

$\mathrm{Res}(f,0)$ est le coefficient de $\frac{1}{z}$

$$f(z) = \frac{1}{z} \sum_{k=0}^{2m} C_{2m}^{k} z^{2k-2m}$$

$$f(z) = \frac{1}{z}\left[\sum_{k=0}^{m-1} C_{2m}^{2k-2m} + C_{2m}^{m} z^{2m-2m} + \sum_{k=m+1}^{2m} C_{2m}^{k} z^{2k-2m}\right]$$

$$f(z) = \frac{1}{z}\left(\sum_{k=0}^{m-1} C_{2m}^{k} z^{2k-2m} + C_{2m}^{m} + \sum_{k=m+1}^{2m} C_{2m}^{k} z^{2k-2m}\right)$$

$$= \sum_{k=0}^{m-1} C_{2m}^{k} z^{2k-2m-1} + \frac{C_{2m}^{m}}{z} + \sum_{k=m+1}^{2m} C_{2m}^{k} z^{2k-2m-1}$$

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$$0 \leqslant k \leqslant n-1 \Rightarrow 0 \leqslant 2k \leqslant 2(n-1) = 2n-2$$

$$\Rightarrow 2k - 2n - 1 \leqslant -3$$

$$n+1 \leqslant k \leqslant 2n \Rightarrow 2k \geqslant 2n+2$$

$$\Rightarrow 2k - 2n - 1 \geqslant 1$$

Le coefficient de $\frac{1}{z}$ est $C_{2n}^{n} = C_{-1}$

donc $\operatorname{Res}(f,0) = C_{2n}^{n} = \dfrac{(2n)!}{(n!)^2}$

$$\int_{C(0,1)} f(z)\,dz = 2\pi i \frac{(2n)!}{(n!)^2}$$

$$I_n = \frac{-i}{2^{2n}} \int_{C(0,1)} f(z)\,dz = \frac{-i}{2^{2n}} 2\pi i \frac{(2n)!}{(n!)^2}$$

$$= \frac{-i}{2^{2n}} 2\pi i \frac{(2n)!}{(n!)^2} = 2\pi \times \frac{(2n)!}{2^{2n}(n!)^2}$$

$$\int_0^{2\pi} (\cos t)^{2n}\,dt = 2\pi \frac{(2n)!}{2^{2n}(n!)^2}$$

Exercice 9 :

$$I = \int_0^{+\infty} \frac{\sin^2 x}{x^2}\,dx.$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos 2x = \frac{e^{i2x} + e^{-2ix}}{2}$$

$$1 - \cos 2x = \frac{2 - e^{2ix} - e^{-2ix}}{2}$$

$x = -t$

$dx = -dt$

$$1 - \cos 2x = \frac{1 - e^{2ix}}{2} + \frac{1 - e^{-2ix}}{2}$$

$$\sin^2 x = \frac{1}{4}\left(1 - e^{2ix} + 1 - e^{-2ix}\right)$$

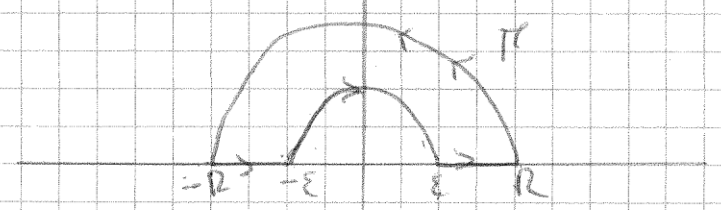
$$\frac{\sin^2 x}{x^2} = \frac{1}{4}\left(\frac{1 - e^{2xi}}{x^2} + \frac{1 - e^{-2xi}}{x^2}\right)$$

$$\int_0^{+\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{4}\left(\int_0^{+\infty} \frac{1 - e^{2xi}}{x^2}\,dx + \int_0^{+\infty} \frac{1 - e^{-2xi}}{x^2}\,dx\right)$$

$$\int_0^{+\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{4}\left(\int_0^{+\infty} \frac{1 - e^{2xi}}{x^2}\,dx + \int_{-\infty}^{0} \frac{1 - e^{2ti}}{t^2}\,dt\right)$$

$$\int_0^{+\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{4}\int_{-\infty}^{+\infty} \frac{1 - e^{2xi}}{x^2}\,dx$$

soit $f(z) = \dfrac{1 - e^{2zi}}{z^2}$



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soit $\Omega = D(0, \rho)$ avec $0 < \varepsilon < R < \rho$

$$f : \Omega \setminus \{0\} \longrightarrow \mathbb{C}$$
$$z \longrightarrow \frac{1 - e^{2iz}}{z^2}$$ est holomorphe

sur $\Omega - \{0\}$, $\Gamma \subset \Omega - \{0\}$, Γ est fermé,

simple de classe C^1 par morceaux

Le théorème des Résidus $\Rightarrow \int_{\Gamma} f(z)\,dz = 0$

car $0 \notin \operatorname{Int} \Gamma$

d'un autre côté,

$$\int_{\Gamma} f(z)\,dz = \int_{[-R,-\varepsilon]} f(z)\,dz + \int_{C_\varepsilon^+} f(z)\,dz + \int_{[\varepsilon,R]} f(z)\,dz + \int_{C_R} f(z)\,dz$$

$$\Rightarrow \int_{[-R,-\varepsilon]} f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{1-e^{2it}}{t^2}\,dt = \int_{\varepsilon}^{R} \frac{1-e^{-2ix}}{x^2}\,dx$$

$$\int_{[\varepsilon,R]} f(z)\,dz = \int_{\varepsilon}^{R} \frac{1-e^{2ix}}{x^2}\,dx$$

$$\int_{[-R,-\varepsilon]} f(z)\,dz + \int_{[\varepsilon,R]} f(z)\,dz = \int_{\varepsilon}^{R} \left(\frac{1-e^{2ix}}{x^2} + \frac{1-e^{-2ix}}{x^2} \right) dx$$

$$= \int_{\varepsilon}^{R} \frac{2-(e^{2ix}+e^{-2ix})}{x^2}\,dx = \int_{\varepsilon}^{R} \frac{2-2\cos 2x}{x^2}\,dx$$

$$= 2\int_{\varepsilon}^{R} \frac{1-\cos 2x}{x^2}\,dx = 4\int_{\varepsilon}^{R} \frac{\sin^2 x}{x^2}\,dx$$

on fait tendre $\varepsilon \longrightarrow 0$ et $R \longrightarrow +\infty$

$$4\int_{\varepsilon}^{R} \frac{\sin^2 x}{x^2}\,dx - \int_{C_\varepsilon^+} f(z)\,dz + \int_{C_R} f(z)\,dz = 0$$

$$\lim_{\varepsilon \to 0^+} \int_{C_\varepsilon^+} f(z)\, dz.$$

f est holomorphe sur $D(0,\varepsilon) \setminus \{0\}$

$z_0 = 0$ est un pôle simple de f.

Alors $\displaystyle\lim_{\varepsilon \to 0^+} \int_{C_\varepsilon^+} f(z)dz = i\pi \operatorname{Res}(f,0)$.

$$\lim_{\varepsilon \to 0^+} \int_{C_\varepsilon^+} f(z)\, dz = i\pi(-2i)$$

$$= 2\pi.$$

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**) $\displaystyle\lim_{R \to +\infty} \int_{C_R} f(z)dz.$

$$\int_{C_R} f(z)dz = \int_{C_R} \frac{1-e^{2iz}}{z^2} = \int_{C_R} \frac{dz}{z^2} - \int_{C_R} \frac{e^{2iz}}{z^2}\, dz$$

*) $\displaystyle\lim_{R \to +\infty} \int_{C_R} \frac{dz}{z^2} = 0$ car $h(z) = \frac{1}{z}$ est continue sur $\ell(0,R)$

et vérifie $\displaystyle\lim_{|z| \to +\infty} |z\, h(z)| = \lim_{|z| \to +\infty} \frac{1}{|z|} = 0$

*) $\displaystyle\lim_{R \to +\infty} \int \frac{e^{2iz}}{z^2}\, dz = 0$ car $\dfrac{e^{2iz}}{z^2} = g(z)e^{2iz}$

$= g(z)e^{idz}$

avec $d = 2 > 0$

$g(z) = \dfrac{1}{z^2}$ est holomorphe sur le demi-plan supérieur

et $\displaystyle\lim_{|z| \to +\infty} |g(z)| = \lim_{|z| \to +\infty} \frac{1}{|z|^2} = 0$

D'où $\displaystyle\lim_{R \to +\infty} \int_{C_R} f(z)\, dz = 0$

En faisant $\varepsilon \to 0^+$ et $R \to +\infty$

dans ⊛, on obtient

$$4\int_0^{+\infty} \frac{\sin^2 x}{x^2}\,dx - 2\pi + 0 = 0$$

$$\int_0^{+\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{2\pi}{4} = \frac{\pi}{2}$$

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Exercice 8 :

$$J = \int_0^{+\infty} \frac{x}{x^8+1}\,dx.$$

Considérons : $f(z) = \dfrac{z}{z^8+1}$

Points singulier de f.

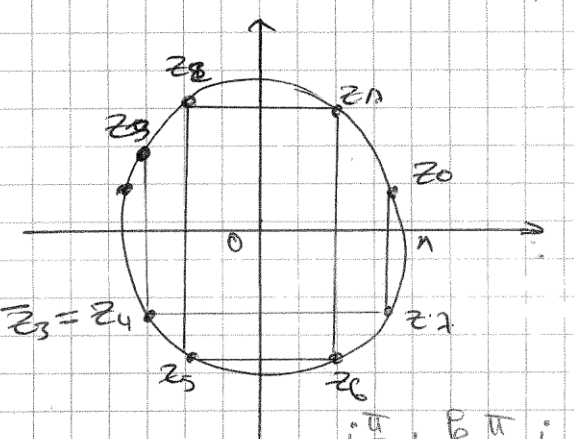
$$D_f = \{ z \in \mathbb{C} \,/\, z^8+1 \neq 0 \}$$

$$z^8+1 = 0 \Rightarrow z^8 = -1 = e^{i\pi}$$

$$z_k = e^{\frac{i\pi}{8} + \frac{2k\pi i}{8}} \qquad 0 \leq k \leq 7$$

$$\Rightarrow z_k = e^{i\frac{\pi}{8} + k\frac{\pi}{4} i} \qquad 0 \leq k \leq 7.$$

$$|z_k| = 1 \qquad \forall\, k = 0, \ldots, 7.$$



$$D_f = \mathbb{C} \setminus \left\{ z_k = e^{i\frac{\pi}{8} + k\frac{\pi}{4} i}, \quad 0 \leq k \leq 7 \right\}$$

Les points singuliers de f sont :

$$z_k = e^{i\frac{\pi}{8} + \frac{k\pi}{4} i} \qquad 0 \leq k \leq 7.$$

Nature :

$$z_k \neq 0 \qquad \forall\, k \in \{0, \ldots, 7\}$$

$$f(z) = \frac{P(z)}{Q(z)}, \quad P(z) = z$$

$$\varphi(z) = z^8 + 1$$

$$P(z_k) = z_k \neq 0 \quad , \quad \varphi(z) = z_k^8 + 1 = 0$$

$$\varphi'(z) = 8z^7 \quad , \quad \varphi'(z_k) = 8z_k^7 \neq 0 \quad \forall k.$$

$$\lim_{z \to z_k} (z - z_k) f(z) = \lim_{z \to z_k} (z - z_k) \frac{P(z)}{\varphi(z)}$$

$$= \lim_{z \to z_k} (z - z_k) \times \frac{P(z)}{\varphi(z) - \varphi(z_k)}$$

$$= \lim_{z \to z_k} \frac{P(z)}{\dfrac{\varphi(z) - \varphi(z_k)}{z - z_k}} = \lim_{z \to z_k} \frac{P(z)}{\varphi'(z_k)} \neq 0 = \frac{P(z_k)}{\varphi'(z_k)}$$

donc z_k est un pôle d'ordre $\overset{m}{\overset{\shortparallel}{1}}$, par suite

$$\mathrm{Res}(f, z_k) = \frac{1}{(1-1)!} \lim_{z \to z_k} \frac{d^{1-1}}{dz^{1-1}} \left((z - z_k)^1 f(z) \right)$$

$$\mathrm{Res}(f, z_k) = (z - z_k) f(z)$$

$$= \lim_{z \to z_k} (z - z_k) f(z) = \frac{P(z_k)}{\varphi'(z_k)} = \frac{z_k}{8 z_k^7}$$

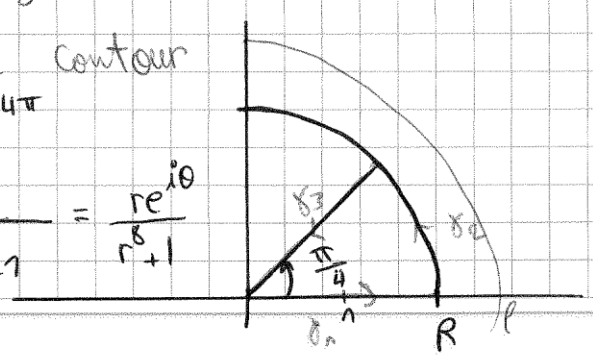
$$= \frac{z_k^3}{8 z_k^8} \qquad (z_k^8 = -1)$$

$$\mathrm{Res}(f, z_k) = -\frac{z_k^3}{8} = -\frac{1}{8} e^{i\frac{\pi}{4} + \frac{k\pi}{2} i}$$

on considère le contour

$$\frac{re^{i\theta}}{(re^{i\theta})^8 + 1} = \frac{r e^{i\theta}}{r^8 \underbrace{e^{i8\theta}}_{e^{i2\pi}} + 1} = \frac{re^{i\theta}}{r^8 + 1}$$

$$\left(re^{i\frac{\pi}{2}}\right)^8 = r^8 e^{i4\pi}$$



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$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ chemin fermé simple, de classe C^1 par morceaux et ne passe pas par les points singuliers de f.

soit $\Omega = D(0,\rho)$ avec $\rho > R > 1$.

$$f : \Omega \setminus \{z_0, z_1, \dots, z_7\} \longrightarrow \mathbb{C}$$
$$z \longrightarrow \frac{z}{z^8+1}$$

f est holomorphe.

$\Gamma \subset \Omega$ le théorème des résidus $\Rightarrow$

$$\int_\Gamma f(z)\,dz = 2\pi i \operatorname{Res}(f, z_0).$$

Car z_0 est le seul point singulier de f qui est à l'intérieur de Γ.

$z_0 \in \operatorname{Int}\Gamma$, $z_k \notin \operatorname{Int}\Gamma$ $1 \leq k \leq 7$.

$$\operatorname{Res}(f, z_0) = -\frac{1}{8} e^{i\frac{\pi}{4}}$$

$$\Rightarrow \int_\Gamma f(z)\,dz = -\frac{2\pi i}{8} e^{i\frac{\pi}{4}} = -\frac{\pi}{4} i e^{i\frac{\pi}{4}} \quad \forall R > 1.$$

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz.$$

*) $\gamma_1 : [0, R] \longrightarrow \mathbb{C}$
$t \longrightarrow t$

$$\int_{\gamma_1} f(z)\,dz = \int_0^R f \circ \gamma_1(t)\, \gamma_1'(t)\,dt = \int_0^R \frac{t}{t^8+1}\,dt$$

$$\lim_{R\to+\infty}\int_{\gamma_n} f(z)dz = \int_0^{+\infty}\frac{t}{t^8+1}\,dt = J$$

$**)$ $\gamma_2 : \left[0,\frac{\pi}{4}\right] \longrightarrow \mathbb{C}$

$\theta \longrightarrow Re^{i\theta}$

$f(z) = \dfrac{z}{z^8+1}$ est continue sur $C(0,R)$ avec $R>1$

$$\lim_{|z|\to+\infty}\left|z f(z)\right| = \lim_{|z|\to+\infty}\frac{|z|^2}{|z^8+1|} = 0$$

γ_2 est un arc de cercle de centre 0 et de rayon R, d'angle $\frac{\pi}{4}$.

Le lemme 2 $\Rightarrow \lim_{R\to+\infty}\int_{\gamma_2} f(z)\,dz = 0$

on peut aussi calculer directement :

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leqslant \ell(\gamma_2)\sup\left|f\circ\gamma_2(\theta)\right| \qquad \theta\in\left[0,\frac{\pi}{4}\right]$$

$$\leqslant \frac{\pi}{4}R\,\frac{R}{R^8-1} \xrightarrow[R\to+\infty]{} 0$$

$***)$ $\gamma_3 : [0,R] \longrightarrow \mathbb{C}$

$t \longrightarrow te^{i\frac{\pi}{4}}$

$$\int_{\gamma_3} f(z)\,dz = -\int_{\gamma_3^-} f(z)\,dz = -\int_0^R f\left(te^{i\frac{\pi}{4}}\right)e^{i\frac{\pi}{4}}\,dt$$

$$= -\int_0^R \frac{te^{i\frac{\pi}{4}}}{\left(te^{i\frac{\pi}{4}}\right)^8+1}\,e^{i\frac{\pi}{4}}\,dt$$

$$= -e^{i\frac{\pi}{2}}\int_0^R \frac{t\,dt}{t^8e^{8\pi i}+1}$$

$$\int_{\gamma_3} f(z)\,dz = -i\int_0^R \frac{t}{t^3+1}\,dt.$$

$$\lim_{R\to+\infty}\int_{\gamma_3} f(z)\,dz = -i\int_0^{+\infty}\frac{t}{t^3+1}\,dt = -i\,J.$$

Faisons tendre $R \to +\infty$ dans :

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = -\frac{\pi}{4}\,i\,e^{i\frac{\pi}{4}}$$

on obtient : $J + 0 - e^{i\frac{\pi}{2}}J = -\frac{\pi}{4}\,i\,e^{i\frac{\pi}{4}}$

$$J\left(1 - e^{i\frac{\pi}{2}}\right) = -\frac{\pi}{4}\,i\,e^{i\frac{\pi}{4}}$$

$$J = \frac{\pi}{4}\,\frac{i\,e^{i\frac{\pi}{4}}}{\left(e^{i\pi/2}-1\right)}$$

$$= \frac{\pi}{2\times 4}\,\frac{i\times 2}{e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}}} = \frac{\pi}{8}\times\frac{1}{\sin\frac{\pi}{4}}$$

$$J = \frac{\pi}{8}\sqrt{2}.$$

iii) $K = \displaystyle\int_0^{+\infty}\frac{dx}{x^3+1}$

$$f(z) = \frac{1}{z^3+1}$$

point singulier de f.

$$z^3+1 = 0 \iff z^3 = -1 = e^{i\pi}$$

$$z_k = e^{i\frac{\pi}{3} + \frac{2k\pi}{3}i}, \quad k = 0, 1, 2$$

$z_0 = e^{i\frac{\pi}{3}}$,

$z_1 = e^{i\pi} = -1$, $z_2 = e^{i\frac{\pi}{3} + \frac{4\pi}{3} i}$

les points singuliers de f sont :

$z_0 = e^{i\frac{\pi}{3}}$, $z_1 = -1$

$z_2 = e^{i\frac{\pi}{3} + \frac{4}{3}\pi i}$

Nature :

$P(z) = 1$, $Q(z) = z^3 + 1$

$Q(z_k) = z_k^3 + 1 = 0$

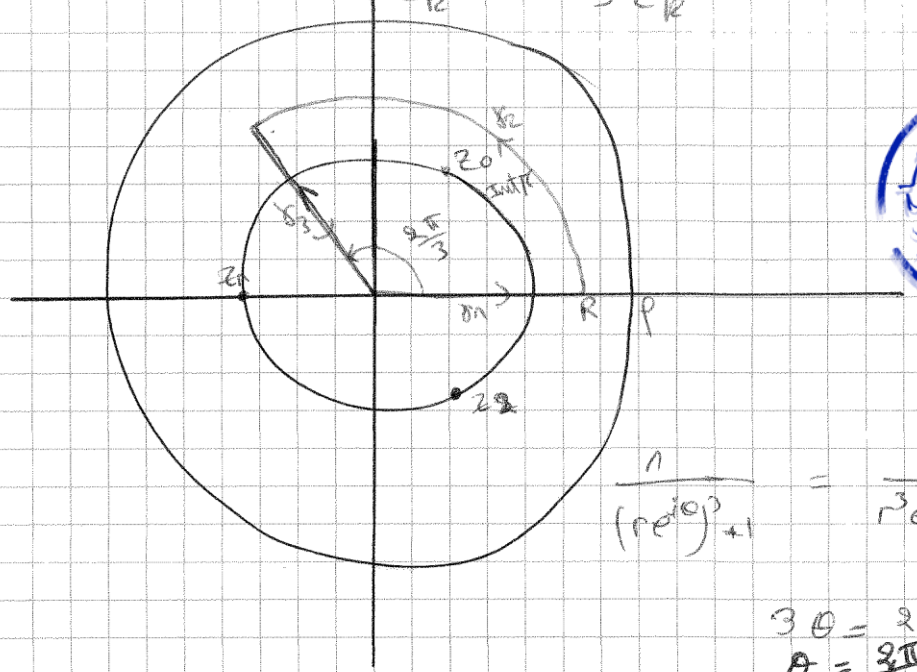
$Q'(z) = 3z^2$

$\Rightarrow Q'(z_k) = 3z_k^2 \neq 0 \quad \forall k = 0, 1, 2$

$$\lim_{z \to z_k} (z - z_k) = \frac{P(z_k)}{Q'(z_k)} = \frac{1}{3z_k^2} \neq 0 \ , \ k = 0, 1, 2$$

z_k est un pôle simple de f d'ordre 1. $(m = 1)$.

$$\mathrm{Res}(f, z_k) = \frac{1}{3z_k^2} = \frac{z_k}{3z_k^3} = -\frac{z_k}{3}$$



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$$\frac{1}{(re^{i\theta})^3 + 1} = \frac{1}{r^3 e^{3i\theta} + 1} = \frac{1}{r^3 + 1}$$

$3\theta = 2\pi$.

$\theta = \frac{2\pi}{3}$.

$\Omega = D(0,\rho)$, $\rho > R > 1$

$$f : \Omega \setminus \{z_0, z_1, z_2\} \longrightarrow \mathbb{C}$$
$$z \longrightarrow \frac{1}{z^3+1}$$

est holomorphe

$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \subset \Omega \setminus \{z_0, z_1, z_2\}$

Γ est fermé, simple, de classe C^1 par morçeaux. Le theorème des résidus $\Rightarrow$

$$\int_\Gamma f(z)\,dz = 2\pi i \operatorname{Res}(f, z_0) \quad \text{car } z_0 \in \operatorname{Int}\Gamma,\ z_1, z_2 \notin \operatorname{Int}\Gamma$$

$\operatorname{Res}(f, z_0) = -\frac{z_0}{3}$, donc $\displaystyle\int_\Gamma f(z)\,dz = -\frac{2\pi i}{3} z_0 = \frac{2\pi i}{3} e^{i\frac{\pi}{3}}$ $\forall R > 1$

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$$

*) $\gamma_1 : [0, R] \longrightarrow \mathbb{C}$
$t \longrightarrow t$

$$\int_{\gamma_1} f(z)\,dz = \int_0^R f\circ\gamma(t)\,\gamma'(t)\,dt = \int_0^R \frac{dt}{t^3+1} \xrightarrow{R\to+\infty} \int_0^{+\infty} \frac{dt}{t^3+1} = K$$

**) $\gamma_2 : \left[0, \frac{2\pi}{3}\right] \longrightarrow \mathbb{C}$
$\theta \longrightarrow Re^{i\theta}$

$f(z) = \frac{1}{z^3+1}$ est continue sur le cercle $C(0,R)$, $\forall R > 1$

$$\lim_{|z|\to+\infty} |z f(z)| = \lim_{|z|\to+\infty} \frac{|z|}{|z^3+1|} = 0 .$$

***) $\gamma_3 : [0,R] \longrightarrow$

$$t \longrightarrow t e^{\frac{2\pi}{3} i}$$

$$\int_{\gamma_3} f(z)\,dz = -\int_{\gamma_3^-} f(z)\,dz = -\int_0^R \frac{e^{\frac{2\pi}{3} i}\,dt}{\left(t e^{\frac{2\pi}{3} i}\right)^3 + 1}$$

$$= -e^{\frac{2\pi}{3} i} \int_0^R \frac{dt}{t^3 e^{2\pi i} + 1} = -e^{2\pi i} \int_0^R \frac{dt}{t^3 + 1}$$

$$\lim_{R \to +\infty} \int_{\gamma_3} f(z)\,dz = -e^{\frac{2\pi}{3} i} K.$$

Faisons tendre $R \longrightarrow +\infty$ dans.

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = -\frac{2\pi i}{3} e^{i\frac{\pi}{3}}$$

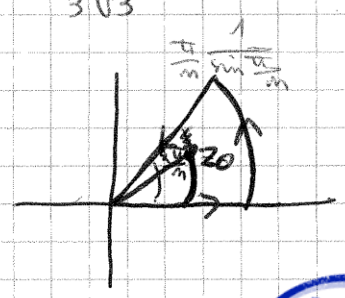
on obtient

$$K + 0 - e^{\frac{2\pi}{3} i} K = \frac{-2\pi i}{3} e^{i\frac{\pi}{3}}$$

$$\Rightarrow \quad K = \frac{\frac{2\pi}{3} i\, e^{i\frac{\pi}{3}}}{e^{\frac{2\pi}{3} i} - 1} = \frac{\pi}{3} \frac{2i}{e^{i\frac{\pi}{3}} - e^{-i\frac{\pi}{3}}}$$

$$\Rightarrow \quad K = \frac{\pi}{3} \times \frac{1}{\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}$$

$$\int_0^{+\infty} \frac{dx}{x^n + 1}, \quad n \geq 2$$



$e^{i\frac{\pi}{n}} = z_0$

$$z^4 + (1-i)z^3 - iz^2 =$$
$$= z^2(z^2 + (1-i)z - i)$$
$$= z^2\left((z+1)(z-i)\right).$$

$e^{i\frac{\pi}{n}} = z_0$

$$I_n = \frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}$$

Exercice 8 :

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+4)} = 2\pi i \left(\mathrm{Res}(f, i) + \mathrm{Res}(f, 2i) \right).$$

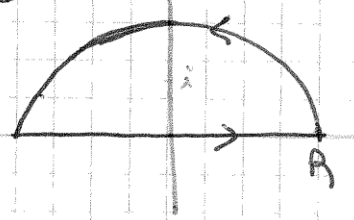
$$f(z) = \frac{1}{(z^2+1)(z^2+4)}$$

Exercice 9 :

$$I = \int_0^{+\infty} \frac{x \sin x}{(x^2+1)^2} dx \Rightarrow I = \int_0^{+\infty} \frac{x(e^{ix} - e^{-ix})}{2i(x^2+1)^2} dx = \frac{1}{2i} \int_0^{+\infty} \frac{x e^{ix}}{(x^2+1)^2} dx - \frac{1}{2i} \int_0^{+\infty} \frac{x e^{-ix}}{(x^2+1)^2} dx$$

$$= \frac{1}{2i} \int_0^{+\infty} \frac{t e^{it}}{(t^2+1)^2} dt + \frac{1}{2i} \int_{-\infty}^{0} \frac{t e^{it}}{(t^2+1)^2} dt$$

$$= \frac{1}{2i} \int_{-\infty}^{+\infty} \frac{t e^{it}}{(t^2+1)^2} dt \quad f(z) = \frac{z e^{iz}}{(z^2+1)^2} = f(z).$$



$$g(z) = (z-i)^2 f(z)$$

$$= \frac{z e^{iz}}{(z+i)^2}.$$

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